

## Research Article

## An Error Efficient Triplet Quadrature Rule

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## Abstract

An anti-Simpson rule has been constructed by using the concept of D. P. Laurie. Furthermore, using the concept of generalized quadrature rule, a triplet quadrature rule  $GQ(f)$  of higher precision has been constructed successfully. The constructed anti-Simpson rule, Simpson's 1/3rd rule and Gaussian quadrature rules of lower precision has been used to formulate the new rule  $GQ(f)$ . The efficiency of new quadrature rule is shown theoretically in error analysis. The numerical supremacy of the constructed rule over its base rules has been reflected in table and figures by numerically evaluating some test integrals.

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## 1. Introduction

In numerical analysis, quadrature rule has been used for numerical evaluation of definite integrals by finite linear combination of weights  $\omega'_i$ s and values of function  $f(\xi)$  at nodes  $\xi'_i$ s. Mathematically,

$$I(f) = \int_a^b f(\xi) d\xi \approx \sum_{i=1}^n \omega_i f(\xi_i) d\xi$$

In literature, Newton-cote type rule and Gaussian type rules are two type of classical rules, which has been used most commonly.

The anti-quadrature rule was constructed by D.P.Laurie [1] in 1996. He adopted Gaussian quadrature rule to construct a sub optimal anti-Gaussian quadrature rule. It was an alternate approach to numerical evaluation definite integral  $\int_{-1}^1 f(\xi) d\xi$ .

The concept of mixed quadrature rules was introduced by R. N. Das and G. Pradhan in 1996 to overcome the difficulties associated with classical techniques for improving the precision of quadrature rules, such as Richardson extrapolation [2, 3]. One of the main advantages of this approach is that it is comparatively easier to implement. In addition, the mixed rule formulation allows the nodes of the existing quadrature rules to be retained, which is often useful in computations.

Later, in 2022, Mohanty and Dash extended this idea by proposing a generalized mixed quadrature technique, where a finite number of lower-order quadrature rules are combined to obtain a rule of higher precision. More recently, Nayak et al. [4] developed a mixed anti-Gaussian quadrature rule for the numerical evaluation of Cauchy principal value integrals. Several other authors [3, 5–9] have also used the idea of mixed quadrature rules to construct higher-order quadrature formulas.

In this paper, an anti-Simpson 3-point rule has been constructed. Furthermore by using the concept of the generalized mixed quadrature rule, a higher-precision triplet quadrature rule is obtained through a suitable combination of the three lower-precision rules. The error has been analysed and the theoretical prediction of the rule has been verified numerically by using random test integrals.

## 2. Development of Anti-Simpson's Rule

Adopting the principle of Laurie [1], an anti-Simpson's 3-point rule  $AS_3(f)$  is to be constructed in such a way that the magnitude of error associated with anti-Simpson's rule is equal to -1 times of the magnitude of error associated with Simpson's  $\frac{1}{3}rd$  rule, by integrating polynomials of degree upto 5. Mathematically,

$$I(f) - AS_3(f) = -[I(f) - SM_3(f)]$$

$$\Rightarrow AS_3(f) = 2I(f) - SM_3(f) \quad (1)$$

where,

$$AS_3(f) = \omega_1 f(\xi_1) + \omega_2 f(\xi_2) + \omega_3 f(\xi_3) \quad (2)$$

The Simpson's  $\frac{1}{3}rd$  rule [10, 11] for numerical approximation of  $\int_{-1}^1 f(\xi) d\xi$  is given by

$$SM_3(f) = \frac{1}{3} [f(-1) + 4f(0) + f(1)]$$

By choosing the polynomials  $1, \xi, \xi^2, \xi^3, \xi^4, \xi^5$  and by using them in (1), we have the following equations.

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 &= 2 \\ \omega_1 \xi_1 + \omega_2 \xi_2 + \omega_3 \xi_3 &= 0 \\ \omega_1 \xi_1^2 + \omega_2 \xi_2^2 + \omega_3 \xi_3^2 &= \frac{2}{3} \\ \omega_1 \xi_1^3 + \omega_2 \xi_2^3 + \omega_3 \xi_3^3 &= 0 \\ \omega_1 \xi_1^4 + \omega_2 \xi_2^4 + \omega_3 \xi_3^4 &= \frac{2}{15} \\ \omega_1 \xi_1^5 + \omega_2 \xi_2^5 + \omega_3 \xi_3^5 &= 0 \end{aligned} \quad (3)$$

After solving the above system of equations, the value of nodes and weights are given below.

$$\begin{aligned} \xi_1 &= \frac{1}{\sqrt{5}}; \quad \xi_2 = 0; \quad \xi_3 = -\frac{1}{\sqrt{5}} \\ \omega_1 &= \frac{5}{3} = \omega_3; \quad \omega_2 = -\frac{4}{3} \end{aligned}$$

By putting the values of  $\xi_i$ 's and  $\omega_i$ 's in (2), we get

$$AS_3(f) = \frac{5}{3} f\left(\frac{1}{\sqrt{5}}\right) - \frac{4}{3} f(0) + \frac{5}{3} f\left(-\frac{1}{\sqrt{5}}\right) \quad (4)$$

### 2.1. Error Analysis

**Theorem 2.1.** Let  $f(\xi)$  be smooth function in  $[-1, 1]$ , then the error bound associated with  $AS_3(f)$  is given by

$$|EAS_3(f)| \leq \frac{1}{90} |f^{iv}(\eta)|$$

where,  $-1 < \eta < 1$

Proof:

The error corresponding to the rule (4) is given by

$$EAS_3(f) = I(f) - AS_3(f)$$

$$EAS_3(f) = I(f) - \left[ \frac{5}{3} \left\{ f\left(\frac{1}{\sqrt{5}}\right) + f\left(-\frac{1}{\sqrt{5}}\right) \right\} - \frac{4}{3} f(0) \right] \quad (5)$$

and by using Maclaruine series expansion, we have

$$f(\xi) = f(0) + \frac{\xi}{1!} f'(0) + \frac{\xi^2}{2!} f''(0) + \frac{\xi^3}{3!} f'''(0) + \dots \quad (6)$$

$$I(f) = \int_{-1}^1 f(\xi) d\xi = 2f(0) + \frac{1}{3}f''(0) + \frac{1}{60}f^{iv}(0) + \frac{1}{2520}f^{vi}(0) + \dots \tag{7}$$

By putting the values of (6) and (7) in (5), we have

$$EAS_3(f) = \frac{1}{90}f^{iv}(0) + \frac{17}{47250}f^{vi}(0) + \dots \tag{8}$$

$$\Rightarrow |EAS_3(f)| \leq \frac{1}{90} |f^{iv}(\eta)|$$

where,  $-1 < \eta < 1$

The above theorem is established that the degree of precision of  $AS_3(f)$  is 3.

### 3. Construction of Triplet Quadrature Rule

In this section, by utilizing the generalised technique [12], a triplet quadrature rule has been constructed by combining three quadrature rules of lower precision. Before proceeding further, certain definitions and conditions from [12] are recalled below.

#### Definition (Generalized Quadrature Rule) [12]

A quadrature rule of higher precision formed out by using  $n$ -rules of lower precision,  $n \in N$ ,  $n \geq 2$  is known as a generalized quadrature rule.

#### Order of a Generalized Quadrature Rule

The number of Constituent rules that is used to form a Generalized quadrature rule is known as order of the Generalized quadrature rule.

#### SR-Conditions

A set of quadrature rules  $R_1, R_2, \dots, R_n$  are said to satisfy SR-Conditions if The degree of precision of the quadrature rules are in monotonically increasing order.

The degree of precision of first two rules must be equal, that is  $dR_1 = dR_2$ . The degrees precision of 2nd, 3rd, 4th, ... nth quadrature rules are in an arithmetic progression with common difference 2.

Using Generalized technique, the triplet quadrature rule is defined by

$$GQ(f) = \alpha_1 AS_3(f) + \alpha_2 S_3(f) + \alpha_3 G_3(f)$$

such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

here  $G_3(f)$  is the Gauss-Legendre three point rule, which is given by

$$G_3(f) = \frac{5}{9} \left[ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right) \right] + \frac{8}{9}f(0).$$

In expanded form, the error in Gauss-Legendre three point rule is derived as follows such that

$$EG_3(f) = \frac{1}{15750}f^{(vi)}(0) + \frac{1}{992250}f^{(viii)}(0) + \dots \tag{9}$$

In expanded form, the error associated with  $S_3(f)$  rule is derived as follows

$$ES_3(f) = -\frac{1}{90}f^{(iv)}(0) - \frac{1}{1890}f^{(vi)}(0) + \dots \tag{10}$$

By multiplying  $\alpha_1$  in (8),  $\alpha_2$  in (10) and  $\alpha_3$  in (9), we have

$$\left(\frac{\alpha_1}{90} - \frac{\alpha_2}{90}\right)f^{iv}(0) + \left(\frac{17\alpha_1}{47250} - \frac{\alpha_2}{1890} + \frac{\alpha_3}{15750}\right)f^{vi}(0) + \dots = I(f) - GQ(f)$$

By restricting the degree of precision of  $GQ(f)$  to 7, we have

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ \frac{\alpha_1}{90} - \frac{\alpha_2}{90} &= 0 \\ \frac{17\alpha_1}{47250} - \frac{\alpha_2}{1890} + \frac{\alpha_3}{15750} &= 0 \end{aligned} \tag{11}$$

By solving the above system of equations, we obtain

$$GQ(f) = \frac{3}{14}AS_3(f) + \frac{3}{14}S_3(f) + \frac{4}{7}G_3(f) \tag{12}$$

### 3.1. Error Analysis

**Theorem 3.1.** Let  $f(\xi)$  be smooth function in  $[-1,1]$ , then the error bound of  $GQ(f)$  is given by

$$|EGQ(f)| \leq \frac{32}{7875 \times 8!} |f^{viii}(\kappa)|$$

where,  $-1 < \kappa < 1$

The above theorem is established that the degree of precision of  $AS_3(f)$  is 7.

## 4. Numerical Verification

To validate the theoretical findings, a set of test integrals is considered and evaluated numerically. The outcomes are illustrated through tables and graphical representations.

### 4.1. Table

The numerical results obtained using the quadrature rules  $AS_3(f)$ ,  $S_3(f)$ ,  $G_3(f)$  and  $GQ(f)$  are reported in Table 1, 2, 3, 4 and 5. The corresponding truncation errors for each method are also presented for comparison.

**Table 1:** Results for  $I_1 = \int_{-1}^1 e^{\xi} d\xi$

Method	Approximation	Absolute Error
$AS_3(f)$	2.338926058495449	0.01147632879215
$S_3(f)$	2.362053756543490	0.01165136925589
$G_3(f)$	2.350336928680012	0.00006545860759
$GQ(f)$	2.350402491039780	0.00000010375218
Exact Value	2.3504023872876028	

**Table 2:** Results for  $I_2 = \int_{-1}^1 e^{-\xi^2/2} d\xi$

Method	Approximation	Absolute Error
$AS_3(f)$	1.682791393453199	0.02845739033110
$S_3(f)$	1.737687106475089	0.02643832269079
$G_3(f)$	1.712020245201909	0.00077146141761
$GQ(f)$	1.711256961528581	0.00000817774428
Exact Value	1.7112487837842976	

**Table 3:** Results for  $I_3 = \int_1^3 \frac{\sin \xi}{\xi+2} d\xi$

Method	Approximation	Absolute Error
$AS_3(f)$	0.405756367567827	0.00012816835662
$S_3(f)$	0.406003918902318	0.00011938297787
$G_3(f)$	0.405887800550142	0.00000326462569
$GQ(f)$	0.405884553005213	0.00000001708076
Exact Value	0.40588453592445245	

**Table 4:** Results for  $I_4 = \int_2^4 \cos(\log \xi) d\xi$

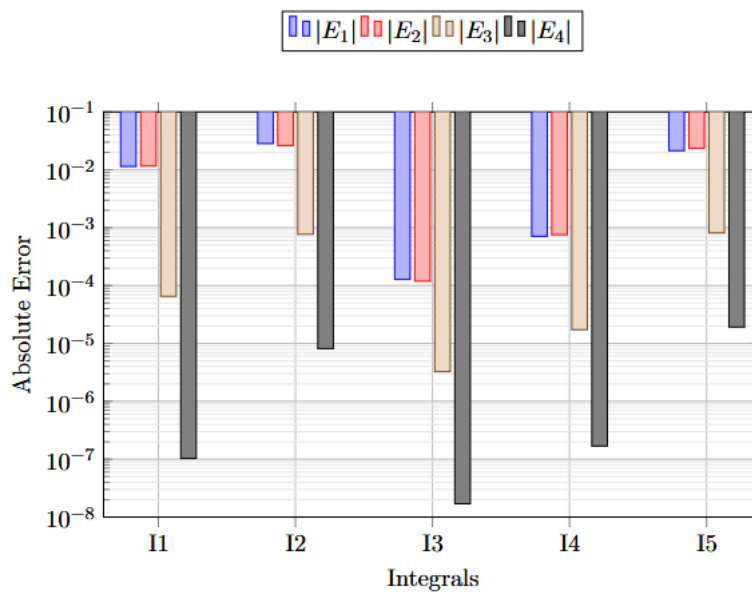
Method	Approximation	Absolute Error
$AS_3(f)$	0.925482941575437	0.00071368894395
$S_3(f)$	0.924008522471238	0.00076073016024
$G_3(f)$	0.924786596849064	0.00001734421758
$GQ(f)$	0.924769083352324	0.00000016927916
Exact Value	0.924769252631483867	

**Table 5:** Results for  $I_5 = \int_{-1}^1 \frac{\xi^2}{\sqrt{2+\xi}} d\xi$

Method	Approximation	Absolute Error
$AS_3(f)$	0.480579503312005	0.02147007060262
$S_3(f)$	0.525783423063209	0.02373384914858
$G_3(f)$	0.501234351634772	0.00081522227985
$GQ(f)$	0.502068828014559	0.00001925409993
Exact Value	0.5020495739146249	

### 4.2. Graphs

For better illumination of the dominance of constructed rule over its base rules, the truncation errors  $E_1, E_2, E_3, E_4$  obtained by the four quadrature rules  $AS_3(f), S_3(f), G_3(f)$  and  $GQ(f)$  upon applying on the test integrals  $I_1$  to  $I_5$  are given in **Figure 1**.



**Figure 1:** Comparison of absolute errors for different quadrature rules

### Observation

From the table and the corresponding figure, the following observations are obtained.

- For  $I_1$ , it is observed that the absolute error associated with  $GQ(f)$  is significantly smaller than that of  $AS_3(f), S_3(f)$  and  $G_3(f)$ . The method  $GQ(f)$  provides accuracy up to nearly 7 significant digits, whereas  $G_3(f)$  gives accuracy up to about 4 significant digits, and both  $AS_3(f)$  and  $S_3(f)$  are accurate only up to 2 significant digits.
- For  $I_2$ , the absolute error of  $GQ(f)$  is again the least among all methods. The accuracy achieved by  $GQ(f)$  is approximately 5 significant digits, while  $G_3(f)$  attains accuracy up to 3 significant digits. The methods  $AS_3(f)$  and  $S_3(f)$  exhibit comparatively larger errors and provide accuracy up to only 1–2 significant digits.
- For  $I_3$ , it is clearly observed that  $GQ(f)$  yields a highly accurate result with accuracy up to about 7 significant digits. The method  $G_3(f)$  also performs well with accuracy up to 5 significant digits, whereas  $AS_3(f)$  and  $S_3(f)$  provide moderate accuracy up to 3 significant digits.
- For  $I_4$ , the method  $GQ(f)$  continues to outperform the other methods, producing accuracy up to nearly 6 significant digits. The  $G_3(f)$  method gives accuracy up to about 4 significant digits, while  $AS_3(f)$  and  $S_3(f)$  show larger errors and achieve accuracy only up to 2 significant digits.
- For  $I_5$ , it is observed that  $GQ(f)$  again provides the smallest error among all methods, yielding accuracy up to 5 significant digits. The method  $G_3(f)$  gives moderate accuracy (around 3 significant digits), whereas  $AS_3(f)$  and  $S_3(f)$  exhibit relatively large errors and lower accuracy.

### 5. Conclusion

From the above tables, graphical representations and observations, it is clearly established that the proposed generalised triplet quadrature rule  $GQ(f)$  outperforms the individual rules  $AS_3(f), S_3(f)$  and  $G_3(f)$  in terms of accuracy. The results demonstrate that  $GQ(f)$  consistently yields the minimum absolute error across all test integrals.

Therefore, it may be concluded that the proposed mixed quadrature rule  $GQ(f)$  provides a more accurate and efficient alternative for the numerical evaluation of definite integrals compared to its base rules.

## Article Information

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