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Research Article

Optimal fourth order iterative method for solving nonlinear equation

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Abstract

The principal objective of this work is to propose a fourth order scheme for solving a nonlinear equation. In terms of computational cost, per iteration, the fourth order method uses two evaluation of the function and one evaluation of first derivative. So the method has satisfied the Kung-Traub optimality conjecture. In addition, the theoretical convergence properties of our schemes are fully explored with the help of the main theorem that demonstrates the convergence order. The performance and effectiveness of our optimal iteration functions have compared with the existing competitors on some standard academic problems.

1. Introduction

One of the most frequent problem in engineering, scientific computing and applied mathematics, in general, is the problem of solving a nonlinear equation $f(x) = 0$. In most of the cases, whenever real problems are faced such as weather forecasting, accurate positioning of satellite systems in the desired orbit, measurement of earthquake magnitudes and other high-level engineering problems, only approximate solutions may get resolved. However, only in rare cases, it is possible to solve the governing equations exactly. The most familiar method of solving non linear equation is Newton's iteration method. The local order of convergence of Newton's method is two and it is an optimal method with two function evaluation per iterative step.

In the past decade, several higher order iterative methods have been developed and analyzed for solving nonlinear equations that improve classical methods such as Newton's method, Chebyshev method, Halley's iteration method, etc. As the order of convergence increases, so does the number of function evaluations per step. Hence, a new index to determine the efficiency called efficiency index is introduced in [1] to measure the balance between these quantities. Kung-Traub [2] conjectured that the order of convergence of any multi-point without memory method with d function evaluations cannot exceed the bound 2^{d-1}, the optimal order. Thus the optimal order for three evaluations per iteration would be four, four evaluations per iteration would be eight and so on. Recently, some fourth and eighth order optimal iterative methods have been developed (see [3–14] and references therein). A more extensive list of references as well as a survey on the progress made in the class of multi-point methods is found in the recent book by Petkovic et al.[11].

This paper is organized as follows: An optimal fourth method are developed by using weight function techniques in section 2. Section 3, convergence order is analyzed. In Section 4, tested numerical examples to compare the proposed methods with other known optimal methods. Section 5 gives concluding remarks.

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2. Construction of proposed methods

We will define an Iterative Function (I.F.) by $x_{n+1} = \psi(x)$. Using the additional information at $x, \phi_1(x), ..., \phi_i(x), i \geq 1$, let x_{n+1} be calculated. Nothing from the past is utilised. Consequently,

$$x_{n+1} = \psi(x, \phi_1(x), ..., \phi_i(x)). \tag{1}$$

A multipoint I.F. without memory is then defined as ψ.

The Newton-Raphson (also known as Newton-I.F.) ($2^{nd}NR$) is provided by

$$\psi_{2^{nd}NR}(x) = x - u(x),\, u(x) = \frac{f(x)}{f'(x)}. \tag{2}$$

With two function evaluations, the $2^{nd}NR$ I.F. is a one-point I.F. that meets the Kung-Traub conjecture for $d = 2$. Also, $EI_{2^{nd}NR} = 1.414$.

2.1. Proposed optimal fourth order I.F

In this way, we attempt to derive a new optimal fourth order I.F.

$$\begin{aligned} \psi_{4^{th}PM}(x) &= \psi_{2^{nd}NR}(x) - H(\tau)\frac{f(\psi_{2^{nd}NR}(x))}{f'(x)}. \\ H(\tau) &= H(0) + \tau\, H'(0) + \frac{1}{2}\,\tau^2 H''(0) + \ldots \text{ and } \tau = \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}. \end{aligned} \tag{3}$$

In the following Theorem we discuss the choice of the parameter $|H''(0)|$ for which the proposed method (3) has the optimal fourth order convergence and we omitted the higher order of weights from $|H''(0)|$.
Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ has continuous derivatives and is suitably smooth. If x_0 is selected in a suitably small neighborhood of x^* and $f(x)$ has a simple root x^* in the open interval D, then the approach (3) has fourth order convergence, when

$$H(0) = 1,\; H'(0) = 2,\; |H''(0)| < \infty \tag{4}$$

The error equation is satisfied.

$$e_{n+1} = \Big(5c_2^3 - c_2c_3)\Big)e^4 + O(e^5). \tag{5}$$

$c_j = \dfrac{f^{(j)}(x^*)}{j!\, f'(x^*)}$, $j = 2,\ 3,\ 4, \ldots$ and $e = x - x^*$.

Proof. Let $\tilde{e} = \psi_{2^{nd}NR}(x) - x^*$, $\hat{e} = \psi_{4^{th}PM}(x) - x^*$. Extending $f(x)$ and $f'(x)$ Around x^* using Taylor's technique, we have

$$f(x) = f'(x^*)\Big(e + c_2\, e^2 + c_3\, e^3 + c_4e^4 + c_5\, e^5 + c_6e^6 + c_7e^7 + c_8e^8 + O(e^9)\Big) \tag{6}$$

and

$$f'(x) = f'(x^*)\Big(1 + 2c_2\, e + 3c_3e^2 + 4c_4e^3 + 5c_5\, e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9\, e^8 + O(e^9)\Big) \tag{7}$$

Thus,

$$\begin{aligned} \tilde{e} &= c_2e^2 + \Big(2c_3 - 2c_2^2\Big)e^3 + \Big(-7c_2c_3 + 4c_2^3 + 3c_4\Big)e^4 + \Big(-8c_2^4 \\ &+ 20\, c_2^2c_3 - 6\, c_3^2 - 10c_2c_4 + 4c_5\Big)\, e^5 + \Big(16c_2^5 - 52\, c_2^3c_3 + 28c_2^2c_4 - 17\, c_3c_4 \\ &+ c_2(33c_3^2 - 13c_5) + 5c_6\Big)\, e^6 - 2\Big(16\, c_2^6 - 64\, c_2^4c_3 - 9\, c_3^3 + 36\, c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 \\ &- 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8\, c_6) - 3c_7\Big)\, e^7 + \Big(64c_2^7 - 304c_2^5c_3 \\ &+ 176c_2^4c_4 + 75c_3^2c_4 + c_2^3(408\, c_3^2 - 92c_5) - 31c_4c_5 - 27c_3c_6 \\ &+ c_2^2(-348\, c_3c_4 + 44c_6) + c_2(-135\, c_3^3 + 64\, c_4^2 + 118\, c_3c_5 - 19c_7) + 7c_8\Big)\, e^8 + \ldots. \end{aligned} \tag{8}$$

Using Taylor's approach, we may expand $f(\psi_{2^{nd}NR}(x))$ about x^* and obtain

$$f(\psi_{2^{nd}NR}(x)) = f'(x^*)\Big(\tilde{e} + c_2\tilde{e}^2 + c_3\tilde{e}^3 + c_4\tilde{e}^4 + O(\tilde{e}^5)\Big) \tag{9}$$

We obtain by simplifying and substituting these equations (6)-(8) and (4) in the (3).

$$\psi_{4^{th}PM}(x) - x^* = \Big(5c_2^3 - c_2c_3)\Big)e^4 + O(e^5).$$

This shows that fourth-order convergence is achieved by the suggested approaches. □

Hence, the proposed methods is

$$\begin{aligned}\psi_{4^{th}PM}(x) &= \psi_{2^{nd}NR}(x) - H(\tau)\frac{f(\psi_{2^{nd}NR}(x))}{f'(x)}.\\ H(\tau) &= 1+2\tau \text{ and } \tau = \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}.\end{aligned} \tag{10}$$

This method (10) has the efficiency $EI_{4thPM} = 1.587$.

3. Numerical Examples

In this section, numerical results on some test functions are compared for the new method $4^{th}PM$ with some existing fourth order methods and Newton's method. Numerical computations have been carried out in the MATLAB software with 500 significant digits. We have used the stopping criteria for the iterative process satisfying $error = |x_N - x_{N-1}| < \varepsilon$, where $\varepsilon = 10^{-50}$ and N is the number of iterations required for convergence. The computational order of convergence is given by ([15])

$$\rho = \frac{\ln|(x_N - x_{N-1})/(x_{N-1} - x_{N-2})|}{\ln|(x_{N-1} - x_{N-2})/(x_{N-2} - x_{N-3})|}.$$

We consider the following iterative methods for solving nonlinear equations for the purpose of comparison: $\psi_{4^{th}SB}$, a method proposed by Sharma et al [16]:

$$y = x - \frac{2f(x)}{3f'(x)}, \psi_{4^{th}SB}(x) = x - \Big(-\frac{1}{2} + \frac{9}{8}\frac{f'(x)}{f'(y)} + \frac{3}{8}\frac{f'(y)}{f'(x)}\Big)\frac{f(x)}{f'(x)}. \tag{11}$$

$\psi_{4^{th}CLND}$, a method proposed by Chun et al [17] :

$$y = x - \frac{2f(x)}{3f'(x)}, \psi_{4^{th}CLND}(x) = x - \frac{16f(x)f'(x)}{-5f'(x)^2 + 30f'(x)f'(y) - 9f'(y)^2}. \tag{12}$$

$\psi_{4^{th}SJ}$, a method proposed by Singh et al [18]:

$$y = x - \frac{2}{3}\frac{f(x)}{f'(x)}, \psi_{4^{th}SJ}(x) = x - \left(\frac{17}{8} - \frac{9}{4}\frac{f'(y)}{f'(x)} + \frac{9}{8}\left(\frac{f'(y)}{f'(x)}\right)^2\right)\left(\frac{7}{4} - \frac{3}{4}\frac{f'(y)}{f'(x_n)}\right)\frac{f(x)}{f'(x)}. \tag{13}$$

The following test functions and their simple zeros for our study are given below [10]:

$$\begin{aligned}
&f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, && x^* = -0.7848959876612125352\ldots\\
&f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, && x^* = -1.2076478271309189270\ldots\\
&f_3(x) = x^3 + 4x^2 - 10, && x^* = 1.3652300134140968457\ldots\\
&f_4(x) = sin(x) + cos(x) + x, && x^* = -0.4566247045676308244\ldots\\
&f_5(x) = \frac{x}{2} - \sin x, && x^* = 1.8954942670339809471\ldots\\
&f_6(x) = x^2 + sin(\frac{x}{5}) - \frac{1}{4}, && x^* = 0.4099920179891371316\ldots
\end{aligned}$$

Table 1, shows that corresponding results for $f_1 - f_6$. We observe that proposed method $4^{th}PM$ is converge in a lesser or equal number of iterations and with least error when compare to compared methods. Note that $4^{th}SB$ and $4^{th}SJ$ methods are getting diverge in f_5 function. Hence, the proposed method $4^{th}PM$ can be considered competent enough to existing other compared equivalent methods.

4. Concluding Remarks

In this work, we established a family of iterative algorithms for solving nonlinear equations that is optimal at the fourth order. Three function evaluations are needed for the approach to obtain an order of convergence of four. The Kung-Traub conjecture is met in the sense of convergence analysis and numerical examples. To demonstrate the superiority of the proposed methods, we have tested a few examples with the proposed scheme and certain recognised schemes. According to the results of the numerical trials, the new techniques could be a useful substitute for solving nonlinear equations.

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Table 1: Numerical results for nonlinear equations.

Methods	$f_1(x), x_0 = -0.9$				$f_2(x), x_0 = -1.6$			
	N	$\|x_1 - x_0\|$	$\|x_N - x_{N-1}\|$	ρ	N	$\|x_1 - x_0\|$	$\|x_N - x_{N-1}\|$	ρ
$2^{nd}NR$ (2)	7	0.1080	7.7326e-74	1.99	9	0.2044	9.2727e-74	1.99
$4^{th}SB$ (11)	4	0.1150	9.7275e-64	3.99	5	0.3343	1.4237e-65	3.99
$4^{th}CLND$ (12)	4	0.1150	1.4296e-64	3.99	5	0.3801	1.1080e-72	3.99
$4^{th}SJ$ (13)	4	0.1150	3.0653e-62	3.99	5	0.3190	9.9781e-56	3.99
$4^{th}PM$ (10)	4	0.1150	5.1046e-68	3.99	5	0.3737	7.4310e-113	4.00
Methods	$f_3(x), x_0 = 0.9$				$f_4(x), x_0 = -1.9$			
$2^{nd}NR$ (2)	8	0.6263	1.3514e-72	2.00	8	1.9529	1.6092e-72	1.99
$4^{th}SB$ (11)	5	0.5018	4.5722e-106	3.99	5	1.5940	6.0381e-92	3.99
$4^{th}CLND$ (12)	5	0.5012	4.7331e-108	3.99	5	1.5894	2.7352e-93	3.99
$4^{th}SJ$ (13)	5	0.4767	3.0351e-135	3.99	5	1.5776	9.5025e-95	3.99
$4^{th}PM$ (10)	5	0.4735	3.2496e-159	3.99	5	1.5519	1.3420e-112	3.99
Methods	$f_5(x), x_0 = 1.2$				$f_6(x), x_0 = 0.8$			
$2^{nd}NR$ (2)	9	2.4123	1.3564e-83	1.99	8	0.3056	3.2094e-72	1.99
$4^{th}SB$ (11)			Diverge		5	0.3801	2.8269e-122	3.99
$4^{th}CLND$ (12)	14	0.0566	6.8760e-134	3.99	5	0.3812	7.8638e-127	3.99
$4^{th}SJ$ (13)			Diverge		5	0.3780	1.4355e-114	3.99
$4^{th}PM$ (10)	6	1.2887	1.5155e-139	3.99	5	0.3840	2.4339e-149	3.99

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