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Research Article

A New Multi - Block Super Class of BDF for Integrating First Order Stiff IVP of ODEs

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Abstract: A new multi-block super class of backward differentiation formula for integrating first order stiff IVPs with a variable mesh size strategy is derived. The proposed scheme approximate two solution values at a time per integration step. The stability properties are achieved by varying the mesh size ratio in the formula to generate more zero stable and A-Stable schemes, capable of solving stiff IVPs of ODEs. Approximates result from the system of stiff ODE problems considered are found to favorably validate the new method in terms of accuracy of the scale error and less executional time in respect to the two methods compared in the study. Hence, the proposed scheme can be used for solving for stiff IVPs of ODEs.

Keywords: block, implicit, IVPs, ordinary differential equation, zero stable

1. Introduction

Numerical Methods play vital role when it comes to solution of differential equation, as some differential equations have no exact solutions at all. Backward differentiation formula came to existence from the work of Curtiss & Hirschfield [1], extended by Cash [2,3], its implicit block method by Ibrahim *et al.*,[4], Super class aspect of BBDF formula by Sueiman *et al.*,[5]; Musa *et al.*,[6 - 8], its diagonally implicit BBDF formula by Zawawi *et al*, [9]. Other work on BBDF include that of Sagir *et al* [10 - 13]. Stiff problem exist in most of Science and Engineering models, solution to this special differential equation became extremely difficult analytically. But, with the aid of implicit numerical scheme, the approximate numerical solution is been achieved. Hence, numerical solutions are use whenever the stiff cases arise for its simplicity and minimum time. A stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small [14]. In search of accurate scheme, scholars keep developing schemes with good accuracy of the scaled error with executional time. The recent work comprises, Soomro *et al* [15], Abdullahi *et al* [16-18], Khan *et al* [19], Shafiq *et al* [20], Rani *et al* [21]. Other methods can be found in [23-31]. These schemes have shown different degree of accuracy and minimum execution time in one problem or the other. It is of paramount advantage to handle Stiff IVPs with a numerical method that possess an A-Stability properties and large stability region, so as to be able to handle any sort of Stiff IVPs, otherwise, the required result would not be obtain.

A differential equation is stiff if it satisfies any or all of the following conditions:

(i) stability requirements in contrast to accuracy constrain the step length,

(ii) some solution components decay much more slowly or rapidly compared to others,

(iii) it has time scales that vary widely, and/or

(iv) All its eigenvalues have negative real parts with large stiffness ratio [22].

2. Materials and Methods

This study considers deriving a super class variable step size BBDF of the form

$$\sum_{j=0}^{3} \alpha_{j,i,r} y_{n+j-1} = h\beta_{k,i,r}(f_{n+k} + \rho f_{n+\frac{2}{k}}) \quad k = 1, 2 \tag{1}$$

The proposed formula (1) can be used for integrating first order stiff IVPs of the form

$$\left.\begin{aligned} y' = f(x,\ \hat{Y}), \quad \widehat{Y}(a) = \varphi\eta, \quad a \le x \le b \\ \text{where } \hat{Y} = (y_1, y_2, y_3, \dots\dots\dots y_n), \eta\bar{\varphi} = (\varphi\eta_1, \varphi\eta_2, \varphi\eta_3, \dots, \varphi\eta_n) \end{aligned}\right\} \tag{2}$$

The linear difference operator L_i associated with (1) is defined as

$$L\{y(x), h\} = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h\beta_j y'(x + jh)\right] \tag{3}$$

Where $y(x)$, a test function and is differential on $[a, b]$

The equation (1) is constructed using a linear operator L_i. For first and second points, define the linear operator L_1 and L_2 associated with (1) as

$$\left.\begin{aligned} L_1[y(x_n), h)]: \alpha_{0,i,r}\, y_{n-1+}\alpha_{1,i}\, y_n + \alpha_{2,i}\, y_{n+1} + \alpha_{3,i}\, y_{n+2} \\ - h\beta_{k,i}[f_{n+1} - \rho f_{n+2}] = 0 \\ L_2[y(x_n), h)]: \alpha_{0,i,r}\, y_{n-1+}\alpha_{1,i}\, y_n + \alpha_{2,i}\, y_{n+1} + \alpha_{3,i}\, y_{n+2} \\ - h\beta_{k,i}[f_{n+2} + f_{n+1}] = 0 \end{aligned}\right\} \tag{4}$$

Case 1&2: $k = i = 1 \& k = i = 2$

The associated relationship for (4) is

$$\left.\begin{aligned} \alpha_{0,1} y(x_n - rh) + \alpha_{1,1} y(x_n) + \alpha_{2,1} y(x_n + h) + \\ \alpha_{3,1} y(x_n + 2h) - h\beta_{1,1}[f(x_n + h) + f(x_n + 2h)] = 0 \\ \alpha_{0,2} y(x_n - rh) + \alpha_{1,2} y(x_n) + \alpha_{2,2} y(x_n + h) + \\ \alpha_{3,2} y(x_n + 2h)) - h\beta_{2,2}[f(x_n + 2h) + f(x_n + h)] = 0 \end{aligned}\right\} \tag{5}$$

Expanding $y(x_n - rh)$, $(x_n), y(x_n + h)$, $y(x_n + 2h)$, $f(x_n + h)$, $f(x_n + 2h)$ and $f(x_n)$ from (5) using Taylor's series expansion about x_n. We obtained the following coefficient for the two cases respectively

$$\left.\begin{aligned} C_{0,1} = \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} = 0 \\ C_{1,1} = -r\alpha_{0,1} + \alpha_{2,1} + 2\alpha_{3,1} - \beta_{1,1}(1 - \rho) = 0 \\ C_{2,1} = \tfrac{1}{2} r^2 \alpha_{0,1} + \tfrac{1}{2}\alpha_{2,1} + 2\alpha_{3,1} - \beta_{1,1}(1 - 2\rho) = 0 \\ C_{3,1} = -\tfrac{1}{6} r^3 \alpha_{0,1} + \tfrac{1}{6}\alpha_{2,1} + \tfrac{4}{3}\alpha_{3,1} - \beta_{1,1}(\tfrac{1}{2} - 2\rho) = 0 \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} C_{0,2} = \alpha_{0,2} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{3,2} = 0 \\ C_{1,2} = -r\alpha_{0,2} + \alpha_{2,2} + 2\alpha_{3,2} - \beta_{2,2}(1 - \rho) = 0 \\ C_{2,2} = \tfrac{1}{2} r^2 \alpha_{0,2} + \tfrac{1}{2}\alpha_{2,2} + 2\alpha_{3,1} - \beta_{2,2}(2 - \rho) = 0 \\ C_{3,2} = -\tfrac{1}{6} r^3 \alpha_{0,2} + \tfrac{1}{6}\alpha_{2,2} + \tfrac{4}{3}\alpha_{3,2} - \beta_{1,1}\left(2 - \tfrac{1}{2}\rho\right) = 0 \end{aligned}\right\} \tag{7}$$

In deriving the first pointy_{n+1} and second point y_{n+2} the coefficient $\alpha_{2,1}$ and $\alpha_{3,2}$ are normalized to 1. Now, solving the simultaneous equation (6) and (7) for $\alpha_{j,i's}$, $\beta_{j,i's}$, arranging and substituting the values in (5), to get the first and second points as

$$y_{n+1} = -\frac{28\rho+5}{r(4\rho r+2r^2+16\rho+5r+4)}y_{n-1} - \frac{7\rho r^3+19\rho r^2-5r^3-8\rho r-10r^2-28\rho-5r-5}{r(4\rho r+2r^2+16\rho+5r+4)}y_n + \frac{7\rho r^2+23\rho r-5r^3-3r^2+8\rho-5r-1}{4\rho r+2r^2+16\rho+5r+4}y_{n+2} + \frac{7r^2+15r+4}{4\rho r+2r^2+16\rho+5r+4}hf_{n+1} - \frac{7r^2+15r+4}{4\rho r+2r^2+16\rho+5r+4}\rho hf_{n+2} \qquad (8)$$

$$y_{n+2} = -\frac{2(\rho+2)}{r(\rho r+\rho-3r-8)(r+1)}y_{n-1} + \frac{\rho r^2+3\rho r+r^2+2\rho+4r+4}{r(\rho r+\rho-3r-8)}y_n - \frac{2(\rho r+2r^2+2\rho+8r+8)}{r(\rho r+\rho-3r-8)(r+1)}y_{n+1} - \frac{2(r+2)}{(\rho r+\rho-3r-8)}hf_{n+2} + \frac{2(r+2)}{(\rho r+\rho-3r-8)}\rho hf_{n+1} \qquad (9)$$

Hence, (8) - (9) is called a new multi – block super class of BDF with variable step size strategy (MBSBDF) for integrating first order stiff IVPs with a variable mesh size strategy. From the proposed scheme, different stable methods can be obtain by appropriate changes in the mesh size ratio r. The newly super class scheme (8) and (9) will consider the value of the free parameter as $\rho = 1/5$, see [7].

Table 1. Variable step size ratios with the stable methods obtained (with $\rho = 1/5$)

Step Size Ratio	Formulae Obtained
$r = 1$	$y_{n+1} = -\frac{53}{75}y_{n-1} + \frac{9}{5}y_n - \frac{7}{75}y_{n+2} + \frac{26}{15}hf_{n+1} - \frac{26}{75}hf_{n+2}$ $y_{n+2} = \frac{11}{53}y_{n-1} - \frac{51}{53}y_n + \frac{93}{53}y_{n+1} + \frac{30}{53}hf_{n+2} - \frac{6}{53}hf_{n+1}$
$r = 2$	$y_{n+1} = -\frac{53}{268}y_{n-1} + \frac{387}{268}y_n - \frac{33}{134}y_{n+2} + \frac{155}{67}hf_{n+1} - \frac{31}{67}hf_{n+2}$ $y_{n+2} = \frac{11}{201}y_{n-1} - \frac{46}{76}y_n + \frac{328}{201}y_{n+1} + \frac{40}{67}hf_{n+2} - \frac{8}{67}hf_{n+1}$
$r = \frac{5}{6}$	$y_{n+1} = -\frac{1431}{1510}y_{n-1} + \frac{1518}{755}y_n - \frac{19}{302}y_{n+2} + \frac{3845}{2416}hf_{n+1} - \frac{769}{2416}hf_{n+2}$ $y_{n+2} = \frac{27}{95}y_{n-1} - \frac{102}{95}y_n + \frac{34}{19}y_{n+1} + \frac{85}{152}hf_{n+2} - \frac{17}{152}hf_{n+1}$

3. Analysis of the Method

3.1. Zero Stability of the proposed formula

In this section, we analyze stability property of the formula (8-9) for different step size ratio r.

Definition According to [5], a linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.

Definition A linear multistep Method is said to be an A-stable method if its stability region covers the entire negative half-plane, [5].

The stability of the method (8-9) can be obtains by using the standard test equation of the form

$$y' = \Lambda y \qquad \Lambda \text{ is a complex number}, \qquad Re(\Lambda) < 0 \qquad (10)$$

$r = 1$

$$\left.\begin{aligned} y_{n+1} &= -\frac{53}{75}y_{n-1} + \frac{9}{5}y_n - \frac{7}{75}y_{n+2} + \frac{26}{15}hf_{n+1} - \frac{26}{75}hf_{n+2} \\ y_{n+2} &= \frac{11}{53}y_{n-1} - \frac{51}{53}y_n + \frac{93}{53}y_{n+1} + \frac{30}{53}hf_{n+2} - \frac{6}{53}hf_{n+1} \end{aligned}\right\} \quad (11)$$

Putting (10) into (11)

$$\left.\begin{aligned} y_{n+1} &= -\frac{53}{75}y_{n-1} + \frac{9}{5}y_n - \frac{7}{75}y_{n+2} + \frac{26}{15}h\lambda y_{n+1} - \frac{26}{75}h\lambda y_{n+2} \\ y_{n+2} &= \frac{11}{53}y_{n-1} - \frac{51}{53}y_n + \frac{93}{53}y_{n+1} + \frac{30}{53}h\lambda_{n+2} - \frac{6}{53}h\lambda y_{n+1} \end{aligned}\right\} \quad (12)$$

Now, (12) can be transform in a matrix form as

$$\begin{bmatrix} 1-\frac{26}{15}h\lambda & \frac{7}{75}+\frac{26}{15}h\lambda \\ -\frac{93}{53}+\frac{6}{53}h\lambda & 1-\frac{30}{53}h\lambda \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{53}{75} & \frac{9}{5} \\ \frac{11}{53} & -\frac{51}{53} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} \quad (13)$$

where

$$A = \begin{bmatrix} 1-\frac{26}{15}h\lambda & \frac{7}{75}+\frac{26}{15}h\lambda \\ -\frac{93}{53}+\frac{6}{53}h\lambda & 1-\frac{30}{53}h\lambda \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} -\frac{53}{75} & \frac{9}{5} \\ \frac{11}{53} & -\frac{51}{53} \end{bmatrix} \quad (14)$$

To find the first characteristic polynomial for (14), we use

det $[At - B] = 0$. To get the polynomial for $r = 1$ as follows

$$R_1(h,t) = \frac{1542}{1325}t^2 - \frac{6764}{3975}t^2h - \frac{1948}{1325}t + \frac{1248}{1325}t^2h^2 - \frac{7124}{3975}th + \frac{406}{1325} \quad (15)$$

Using similar procedure as in above for the step sizes $r = 2$, $r = \frac{1}{2}$ and $r = \frac{5}{6}$ to obtain the following polynomial respectively

$$R_2(h,t) = \frac{6293}{4489}t^2 - \frac{29423}{13467}t^2h - \frac{26191}{17956}t + \frac{5952}{4489}t^2h^2 - \frac{20317}{13467}th + \frac{1019}{17956} \quad (16)$$

$$R_{\frac{5}{6}}(h,t) = \frac{168}{151}t^2 - \frac{36451}{22952}t^2h - \frac{22359}{14345}t + \frac{39219}{45904}t^2h^2 - \frac{44145}{22952}th + \frac{6399}{14345} \quad (17)$$

Put $h = h\lambda$ in (15-17), we have

Set $h = h\lambda = 0$ in (15-17) and solve for t in all the polynomials. The following table is obtained with the respective roots of the polynomials t

Table 2. Zero stability of the proposed formulae

Step Size Ratio (r)	Roots of the proposed methods
$r = 1$	$t = 1, 0.2632944228$
$r = 2$	$t = 1, 0.04048148737$
$r = \frac{5}{6}$	$t = 1, 0.4009398496$

From table 2, it has been shown that the formula (8-9) with the mesh size ratios tested ($r = 1,\ r = 2\ \ and\ r = \frac{5}{6}$) are zero stable methods in accordance with definition 1.

3.2. A - Stability of the proposed formulae

The region for the absolute stability of the new scheme is plotted, by considering the stability polynomials in (15), (16) & (17). The set of point defined by $t = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ describes the boundary of the stability region. The following region was the complex plot of the proposed methods with Maple Software.

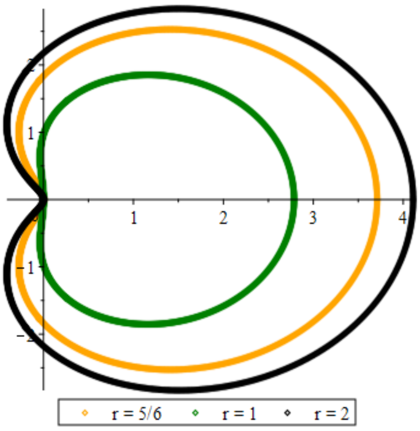

Figure 1. A combined plot for A-Stability regions for $\boldsymbol{r = 1}$, $\boldsymbol{r = 2}$ & $\boldsymbol{r = \frac{5}{6}}$

The region of the absolute stability covered the entire negative left half plane in figure 1; by definition 2, the newly derived formulae are A-Stable methods.

Considering the plotted stability region the method obtained with the step size ratio r = 1 in formula has a larger stability region can be more effective in handling the step cases.

4. Implementation of the Method

In this section Newton's iteration is considered for the implementation of (8–9) across different choice of the mesh size ratio. The method (8 – 9) can be transform to the form:

$$\left.\begin{aligned} y_{n+1} &= \theta_1 y_{n+2} + \gamma_1 h f_{n+1} + \gamma_2 h f_n + v_1 \\ y_{n+2} &= \theta_2 y_{n+1} + \gamma_3 h f_{n+1} + \gamma_4 h f_{n+2} + v_2 \end{aligned}\right\} \quad (18)$$

where v_1 and v_2 are the back values. (18) Will be transforming to the following form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} + h\begin{bmatrix} \gamma_1 & 0 \\ \gamma_3 & \gamma_4 \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (19)$$

Equation (19) can also be written as

$$(I - B)Y = h(C_1 G_1 + C_2 G_2) + \in \quad (20)$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} \gamma_1 & 0 \\ \gamma_3 & \gamma_4 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}, G_2 = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} \quad (21)$$

Let

$$G = (I - B)Y - h(C_1 G_1 + C_2 G_2) - \in = 0 \quad (22)$$

Therefore, the Newton's iteration for the proposed method (RBBDF) is going to be

$$Y_{n+1,n+1}^{(i+1)} - Y_{n+1,n+1}^{(i)} = -\left[G_j'\left(Y_{n+1,n+1}^{(i)}\right)\right]^{-1}\left[G_j\left(Y_{n+1,n+1}^{(i)}\right)\right] \quad (23)$$

Equation (23) is represented as

$$Y_{n+1,n+1}^{(i+1)} - Y_{n+1,n+1}^{(i)} = \left[(I-B) - hC_1\frac{\eth G_1}{\eth Y}\left(Y_{n+1,n+1}^{(i)}\right) - hC_2\frac{\eth G_2}{\eth Y}\left(Y_{n+1,n+1}^{(i)}\right)\right]^{-1} \times \left[(I-B)\left(Y_{n+1,n+1}^{(i)}\right) - hC_1G_1 - hC_2G_2 - \in\right] \quad (24)$$

To compute the MAXE from the proposed algorithm, let $y_i\ and\ y(x_i)$ be the approximate and the exact solution of (2), respectively. The absolute error is written as

$$(error_i)_t = |(y_i)_t - (y(x_i)_t)| \quad (25)$$

The MAXE is given by

$$MAXE = \underbrace{max}_{1\le i\le T}\ (\underbrace{max\ (error_i)_t}_{1\le i\le N}). \quad (26)$$

where T and N are the number of steps and equations respectively.

Let $Y_{n+1}^{(i+1)}$ be the $(i+1)^{th}$ iteration and

$$E_{1,2}^{(i+1)} = Y_{n+1,n+1}^{(i+1)} - Y_{n+1,n+1}^{(i)} \quad (27)$$

From (24) we have

$$\mathrm{E}_{1,2}^{(\mathrm{i}+1)} = \bar{\mathrm{B}}^{-1} - \bar{\mathrm{C}} \quad (28)$$

Now (28) is equivalent to

$$\bar{\mathrm{B}}\mathrm{E}_{1,2}^{(\mathrm{i}+1)} = \bar{\mathrm{C}} \quad (29)$$

where

$$\bar{\mathrm{B}} = [(I-B) - hC_1\frac{\eth G_1}{\eth Y}\left(Y_{n+1,n+1}^{(i)}\right) - hC_2\frac{\eth G_2}{\eth Y}\left(Y_{n+1,n+1}^{(i)}\right)]^{-1} \quad (30)$$

And

$$\bar{\mathrm{C}} = -[(I-B)\left(Y_{n+1,n+1}^{(i)}\right) - hC_1G_1 - hC_2G_2 - \in] \quad (31)$$

Equation (29) Would be solve with Newton's iteration for the different mesh size values of r

$$\bar{\mathrm{B}} = \begin{bmatrix} 1-\gamma_1 h\frac{\eth f_{n+1}}{\eth y_{n+1}} & -\theta_1 \\ -\theta_2 - \gamma_3 h\frac{\eth f_{n+1}}{\eth y_{n+1}} & 1-\gamma_4 h\frac{\eth f_{n+2}}{\eth y_{n+2}} \end{bmatrix},\ \bar{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^i + \theta_1 y_{n+2}^i + \gamma_1 hf_{n+1}^i + \gamma_2 hf_n^i + v_1 \\ -y_{n+2}^i + \theta_2 y_{n+1}^i + \gamma_3 hf_{n+1}^i + \gamma_4 hf_{n+2}^i + v_2 \end{bmatrix}$$

when r = 1

$$\bar{\mathrm{B}} = \begin{bmatrix} 1-\frac{26}{15}h\frac{\eth f_{n+1}}{\eth y_{n+1}} & -\frac{1}{75}-\frac{26}{75}h\frac{\eth f_{n+2}}{\eth y_{n+2}} \\ -\frac{93}{53}-\frac{6}{53}h\frac{\eth f_{n+1}}{\eth y_{n+1}} & 1-\frac{30}{53}h\frac{\eth f_{n+2}}{\eth y_{n+2}} \end{bmatrix},\ \bar{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^i - \frac{7}{75}y_{n+2}^i + \frac{26}{15}hf_{n+1}^i - \frac{6}{75}hf_{n+2}^i + v_1 \\ -y_{n+2}^i + \frac{93}{53}y_{n+1}^i - \frac{6}{53}hf_{n+1}^i + \frac{30}{53}hf_{n+2}^i + v_2 \end{bmatrix}$$

When $r = 2$

$$\bar{\mathrm{B}} = \begin{bmatrix} 1-\frac{155}{67}h\frac{\eth f_{n+1}}{\eth y_{n+1}} & -\frac{33}{134}-\frac{31}{67}h\frac{\eth f_{n+2}}{\eth y_{n+2}} \\ -\frac{328}{201}-\frac{8}{67}h\frac{\eth f_{n+1}}{\eth y_{n+1}} & 1-\frac{40}{67}h\frac{\eth f_{n+2}}{\eth y_{n+2}} \end{bmatrix},$$

$$\bar{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^i - \frac{33}{34}y_{n+2}^i + \frac{155}{67}hf_{n+1}^i - \frac{31}{67}hf_{n+2}^i + v_1 \\ -y_{n+2}^i + \frac{328}{201}y_{n+1}^i - \frac{8}{67}hf_{n+1}^i + \frac{40}{67}hf_{n+2}^i + v_2 \end{bmatrix}$$

When $r = \frac{5}{6}$

$$\bar{\mathrm{B}} = \begin{bmatrix} 1-\frac{384}{2416}h\frac{\eth f_{n+1}}{\eth y_{n+1}} & -\frac{19}{302}-\frac{769}{2416}h\frac{\eth f_{n+2}}{\eth y_{n+2}} \\ -\frac{34}{19}-\frac{17}{152}h\frac{\eth f_{n+1}}{\eth y_{n+1}} & 1-\frac{85}{152}h\frac{\eth f_{n+2}}{\eth y_{n+2}} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} -y_{n+1}^{i} - \frac{19}{302}y_{n+2}^{i} + \frac{3845}{2416}hf_{n+1}^{i} - \frac{769}{2416}hf_{n+2}^{i} + v_1 \\ -y_{n+2}^{i} + \frac{34}{19}y_{n+1}^{i} - \frac{17}{152}hf_{n+1}^{i} + \frac{85}{152}hf_{n+2}^{i} + v_2 \end{bmatrix}$$

5. Result and Discussions

5.1. Numerical Examples

The proposed scheme, a new multi-block super class for integrating first order stiff IVPs developed in (10-11) will be adopted with different step size ratio to solve some IVPs. The computed approximate results would be compared with some existing schemes in the literature to evaluate the performance of the new scheme. Below are some of the problems considered in this study; taken from [4] & [6] for problem 1 and 2 respectively.

Problem 1: $y_1{}'. = -20y_1 - 19y_2 \qquad y_1(0) = 2 \qquad 0 \le x \le 20$
$y_2{}' = -19y_1 - 20y_2 \qquad y_2(0) = 0$

Exact Solution

$y(x) = e^{-5x}$

Problem 2: $y_1'. = 198\mathrm{y}_1 + 199\mathrm{y}_1, \qquad y_1(0) = 10 \le x \le 10$
$y_2' = -398y_1 - 399y_2, \quad y_2(0) = -1$

Exact Solution

$y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$

Below are symbols and notations used in the research with the results of the problems solved in tabular form to depict the comparative differences among the methods compared.

h = step-size

Mtd = Method

Max-err = Maximum Error

NS = Number of Steps

Exec-Time = Execution Time in microseconds

MBSBDF = A new multi-block super class of backward differentiation formula

3NBBDF = 3–point fifth order new BBDF method

3ESBBDF = Extended 3 Point Super Class of Block Backward Differentiation Formula

2BBDF = an implicit r = 2-point BBDF method

Table 3. Approximate results for problem 1 (with different step size ratio r)

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	MBSBDF $(r = 1)$	100	$1.93618(-3)$	$3.81831(-3)$
	MBSBDF $(r = 2)$	100	$1.85552(-3)$	$7.36289(-3)$
	MBSBDF $(r = 5/6)$	100	$1.60623(-3)$	$8.23891(-3)$
10^{-3}	MBSBDF $(r = 1)$	1,000	$1.99937(-5)$	$3.59527(-2)$
	MBSBDF $(r = 2)$	1,000	$3.3017(-4)$	$7.36289(-2)$
	MBSBDF $(r = 5/6)$	1,000	$2.76934(-4)$	$8.23891(-2)$
10^{-4}	MBSBDF $(r = 1)$	10,000	$2.22027(-7)$	$3.26107(-1)$
	MBSBDF $(r = 2)$	10,000	$3.32688(-5)$	$7.36289(-1)$
	MBSBDF $(r = 5/6)$	10,000	$2.79783(-5)$	$8.23891(-1)$
10^{-5}	MBSBDF $(r = 1)$	100,000	$2.48051(-9)$	$2.48051(00)$
	MBSBDF $(r = 2)$	100,000	$3.48440(-7)$	$7.36289(00)$
	MBSBDF $(r = 5/6)$	100,000	$2.82371(-7)$	$8.23891(00)$
10^{-6}	MBSBDF $(r = 1)$	1,000,000	$2.81053(-11)$	$2.81053(+01)$
	MBSBDF $(r = 2)$	1,000,000	$3.65332(-9)$	$7.36289(+01)$
	MBSBDF $(r = 5/6)$	1,000,000	$2.92046(-9)$	$8.23891(+01)$

Table 4. Approximate results for problem 2 (with different step size ratio r)

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	MBSBDF $(r = 1)$	100	$3.26548(-5)$	$4.61073(-5)$
	MBSBDF $(r = 2)$	100	$4.42218(-3)$	$7.36289(-3)$
	MBSBDF $(r = 5/6)$	100	$5.72116(-3)$	$8.23891(-3)$
10^{-3}	MBSBDF $(r = 1)$	1,000	$3.65283(-7)$	$4.67910(-4)$
	MBSBDF $(r = 2)$	1,000	$4.42218(-5)$	$7.36289(-2)$
	MBSBDF $(r = 5/6)$	1,000	$5.72116(-5)$	$8.23891(-2)$
10^{-4}	MBSBDF $(r = 1)$	10,000	$4.70027(-9)$	$4.69213(-3)$
	MBSBDF $(r = 2)$	10,000	$4.42218(-7)$	$7.36289(-1)$
	MBSBDF $(r = 5/6)$	10,000	$5.72116(-7)$	$8.23891(-1)$

10^{-5}	MBSBDF (r = 1)	100,000	4.10002(−11)	5.24421(−2)
	MBSBDF (r = 2)	100,000	4.42218(−9)	7.36289(00)
	MBSBDF (r = 5/6)	100,000	5.72116(−9)	8.23891(00)
10^{-6}	MBSBDF (r = 1)	1,000,000	4.14240(−13)	5.57915(−1)
	MBSBDF (r = 2)	1,000,000	4.42218(−11)	7.36289(+1)
	MBSBDF (r = 5/6)	1,000,000	5.72116(−11)	8.23891(+1)

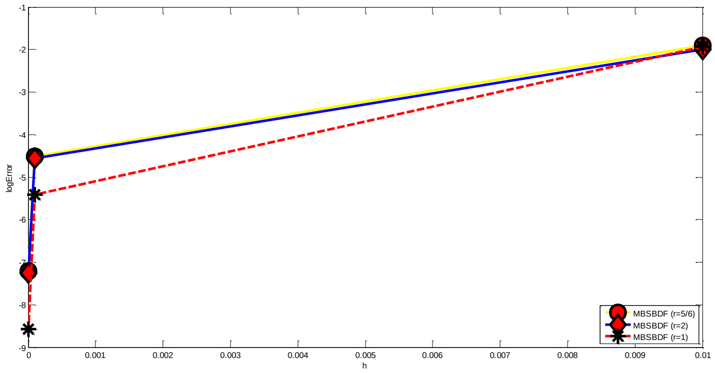

Figure 2. Comparison of$\mathbf{Log_{10}}(\boldsymbol{MAXE})$ against h for problem 1

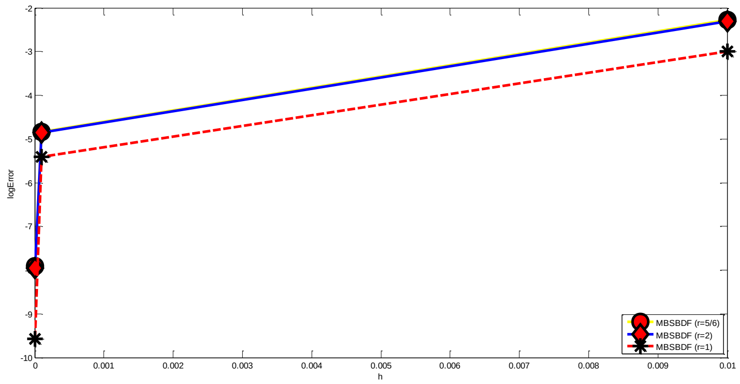

Figure 3. Comparison for $\mathbf{Log_{10}}(\boldsymbol{MAXE})$ against h for problem 2

The result obtained in table 3 & 4 and subsequently the graph in figure 2 & 3, depicts the performance of the newly developed formulae across different choices of the step size ratio r, method with $r = 1$ outperformed those with $r = 2$ and $r = 5/6$ in scale error in problem 1 and compete closely with $r = 2$ computation time in problem 2, the result indicated that the methods with $r = 1$ & $r = 2$ have least scale error compared to the methods $r = 2$ in all the problems considered in the research. The executional time also favors the schemes with $r = 1$ in problem 1 & 2. However, the methods with

$r = 2$ and $r = 5/6$ are closely competing in the execution time. All the schemes have advantages over the scheme with $r = 5/6$ in all the problems solved.

Table 5. Comparison of results for problem 1

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	MBSBDF ($r = 1$)	100	1.93618(−3)	3.81831(−3)
	3NBBDF	666	6.98707(−2)	2.63337(−2)
	2BBDF	500	2.47600(−2)	15,328μs
10^{-3}	MBSBDF ($r = 1$)	1,000	1.99937(−5)	3.59527(−2)
	3NBBDF	6,666	5.40956(−3)	2.60816(−1)
	2BBDF	5,000	2.86614(−3)	127,105 μs
10^{-4}	MBSBDF ($r = 1$)	10,000	2.22027(−7)	3.26107(−1)
	3NBBDF	66,666	3.08942(−5)	2.60725(00)
	2BBDF	55,000	2.90520(−4)	125,5816 μs
10^{-5}	MBSBDF ($r = 1$)	100,000	2.48051(−9)	3.10002(00)
	3NBBDF	666,666	3.18534(−7)	2.60597(01)
	2BBDF	555,555	2.90911(−5)	12,571,049 μs
10^{-6}	MBSBDF ($r = 1$)	1,000,000	2.81053(−11)	3.51201(01)
	3NBBDF	6,666,666	3.19872(−9)	2.60700(2)
	2BBDF	5,555,555	2.90951(−6)	125,811,893 2μs

Table 6. Comparison of results for problem 2

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	MBSBDF ($r = 1$)	100	3.26548(−5)	4.61073(−5)
	3ESBBDF	333	4.58309(−4)	5.54206(−4)
	2BBDF	500	7.18323(−3)	28,413μs
10^{-3}	MBSBDF ($r = 1$)	1,000	3.65283(−7)	4.67910(−4)
	3ESBBDF	3,333	4.70430(−5)	8.52157(−3)
	2BBDF	5,000	7.34012(−4)	256,695μs
10^{-4}	MBSBDF ($r = 1$)	10,000	4.70027(−9)	4.69213(−3)
	3ESBBDF	33,333	4.90784(−7)	4.38708(−2)
	2BBDF	55,000	7.35584(−5)	2,554,368μs
10^{-5}	MBSBDF ($r = 1$)	100,000	4.10002(−11)	5.24421(−2)
	3ESBBDF	333,333	5.70874(−9)	5.36592e(−01)
	2BBDF	555,555	7.35741(−6)	25,625,785μs
10^{-6}	MBSBDF ($r = 1$)	1,000,000	4.14240(−13)	5.57915(−01)
	3ESBBDF	3,333,333	5.77228(−11)	4.37581(00)
	2BBDF	5,555,555	7.35747(−7)	256,394,582μs

Also considering the results in tables 5 and 6 for problems 1 and 2, it have shown that the approximate solution obtained with the new method MBSBDF ($r = 1$) outperformed the 3ESBBDF and 2BBDF in terms of accuracy, computational time in all problems tested, while the accuracy of the approximated solution increases as the step size decreases. However, 3ESBBDF outperformed the 2BBDF in terms of accuracy in problem 2. But 3NSBBDF have advantage in problem 1 over 2BBDF. Similarly, the graphs in Figure 4 and 5 also shows clearly that the scaled errors for the MBSBDF ($r = 1$) is smaller when compared with that in 3ESBBDF and 2BBDF Method in problem 1. While, the 3NBBDF has advantage over 2BBDF in terms of accuracy and execution time in problem 2.

To visibly highlight the performance of the proposed method MBSBDF in relation to the other methods 3NBBDF and 2BBDF. The graphs of *Log10(MAXE)* against h for the problems tested are plotted below.

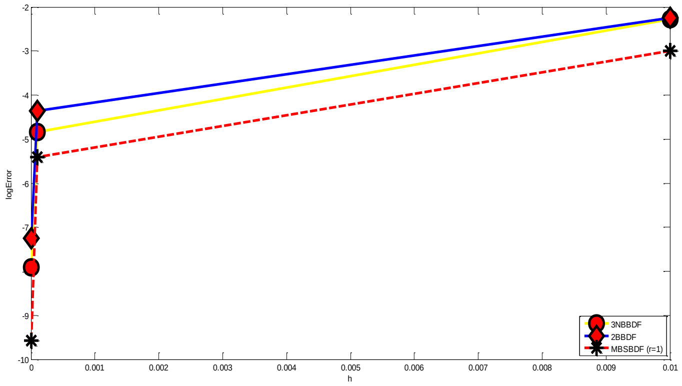

Figure 4. Graph of $\mathbf{Log_{10}}(\boldsymbol{MAXE})$ against h for problem 1

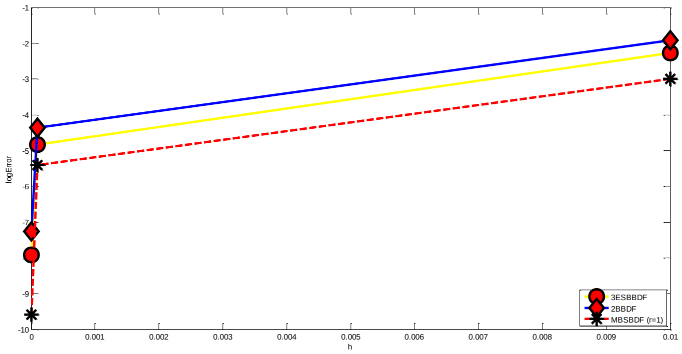

Figure 5. Graph of $\mathbf{Log_{10}}(\boldsymbol{MAXE})$ against h for problem 2

6. Conclusion

A new multi-block super class of backward differentiation formula MBSBDF for solving stiff IVP of ODEs was derived. The proposed methods adopted a variable step size technique and possessed a very good stability. Thus, the new scheme is zero stable and A- stable across different chosen values of the step size ratio of $r = 1, r = 2, \ and\ r = \frac{5}{6}$. The proposed methods solved samples of first order stiff IVPs of ODEs; the results highlighted the performance of the proposed methods in terms of accuracy of the scaled error and executional time compared to two other methods considered in the work. Hence, the proposed method can be used in solving a system of first order stiff IVPs of ODEs.

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