Jagua Publication

Current Research in Interdisciplinary Studies
Vol. 1(5), pages 27-35, year 2022.
www.jpub.org

Higher order multi-point iterative methods for finding GPS User Position

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• Received: 2.12.2022 • Accepted: 22.12.2022 • Published: 28.12.2022

Abstract: The basic operation of Global Positioning System involves the detection and measurement, with a GPS receiver, of data carried on electromagnetic signals transmitted by the earth-orbiting GPS satellite constellation and the computation of the travel time of these received signals. The time measurements are converted to distance measurements, which can be used to compute the unknown position and time of the receiver from the known positions of the satellite transmitters and signal transit times. To solve the problem, at least four satellite's measurements are needed to find user position and receiver time offset. A set of nonlinear navigation equations is formed. These nonlinear equations can be solved using iterative techniques based on newly developed Newton type methods. One of these methods is most efficient and converges faster than other compared iterative methods. A practical study was done to evaluate the new models. El-naggar performance analysis was conducted using data collected by Trimble 4000SSE dual frequency receiver. The results indicate that the improved Newton type methods are simple, fast, and accurate as compared to Newton's method.

Keywords: System of nonlinear equation, Newton's method, Order of convergence, Multistep method, Global Positioning System

1. Introduction

The Global Positioning System (GPS) is all weather and space based navigation system. It is a constellation of a minimum of 24 satellites in near circular orbits, positioned at an approximate height of 20000 km above from the earth. The satellites travel with a velocity of 3.9 km/sec with an orbital period of 11 hours 58 minutes. From the satellite constellation, the equations required for solving the user position conform a nonlinear system of equations. In addition, some practical considerations will be included in these equations. These equations are usually solved through a linearization technique and fixed point iteration method.

That solution of an equation is in a Cartesian coordinated system, and then it is converted into a spherical coordinate system. However, the Earth is not a perfect sphere. Therefore, once the user position is estimated, the shape of the Earth must be taken into consideration. The user position is then translated into the Earth based coordinate system. In this paper, we are going to focus our attention in solving the nonlinear system of equations of the GPS giving the results in a Cartesian coordinate system. The position of a point in space can be found by using the distances measured from this point to some known position in space.

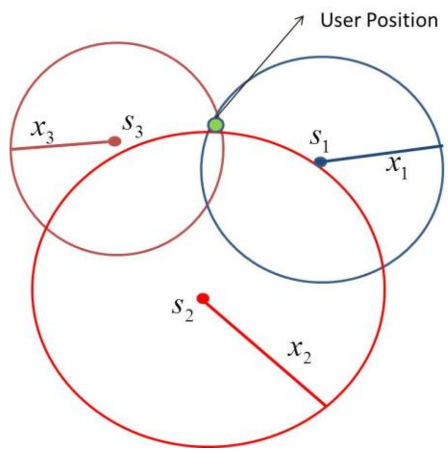

Figure 1: Two dimensional user position

Fig. 1 shows a two dimensional case. In order to determine the user position, three satellites S_1, S_2 and S_3 and three distances are required. The trace of a point with constant distance to a fixed point is a circle in the two-dimensional case. Two satellites and two distances give two possible solutions because two circles intersect at two points. One more circle is needed to uniquely determine the user position. For similar reasons in a three-dimensional case, four satellites and four distances are needed.

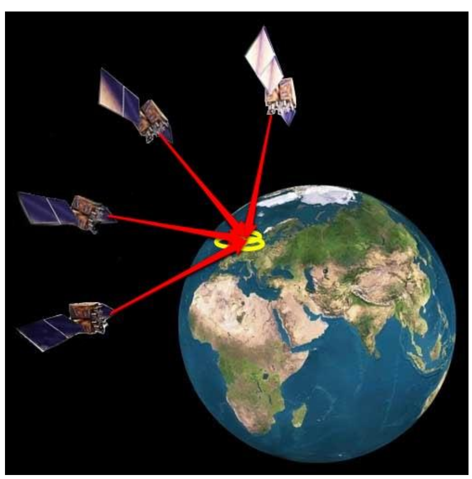

Figure 2: Three dimensional user position

Fig. 2 shows that three dimensional case and it is taken from [5]. The equal-distance trace to a fixed point is a sphere in a three-dimensional case. A GPS receiver knows the location of the satellites because that information is included in the transmitted Ephemeris data. By estimating how far away a satellite is, the receiver also knows it is located somewhere on the surface of an imaginary sphere centered at the satellite. We can find more information about GPS in [1, 11].

2. Basic Equations for Finding User Position

In this section, the basic equations for determining the user position will be presented. Assume that the distance measured is accurate, and under this condition, three satellites should be sufficient. Let us suppose that there are three known points at locations r_1 or (x_1, y_1, z_1), r_2 or (x_2, y_2, z_2), and r_3 or (x_3, y_3, z_3) and an unknown point at r_u or (x_u, y_u, z_u). If the distances between the three known points to the unknown point can be measured as ρ_1, ρ_2, ρ_3 these distances can be written as

$$\begin{aligned} \rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2}, \\ \rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2}, \\ \rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2}. \end{aligned} \tag{1}$$

Because there are three unknowns and three equations, the values of x_u, y_u, z_u can be determined from these equations. Theoretically, there should be two sets of solutions as they are second order equations. These equations can be solved by linearizing them and making an iterative approach. In GPS operation, the positions of the satellites are given. This information can be obtained from the data transmitted from the satellites. The distances from the user to the satellites must be measured simultaneously at a certain time instance. Each satellite transmits a signal with a time reference associated with it. By measuring the time of the signal traveling from the satellite to the user, the distance between the user and the satellite can be found. The distance measurement is discussed in the next section.

3. Measurement of Pseudorange

Every satellite sends a signal at a certain time t_{si}. The receiver will receive the signal at a later time t_u. The distance between the user and the satellite i can be determined as

$$\rho_{iT} = c(t_u - t_{si}), \tag{2}$$

where c is the speed of light, ρ_{iT} is often referred to as the true value of pseudorange from user to satellite, t_{si} is referred to as the true time of transmission from satellites i and t_u is the true time of reception. From a practical point of view, it is difficult, if not impossible, to obtain the correct time from the satellite or the user. The actual satellite clock time t_{si}' and actual user clock time t_u' are related to the true time as

$$t_{si}' = t_{si} + \Delta b_i, \quad t_u' = t_u + b_{ut}, \tag{3}$$

where Δb_i is the satellite clock error and b_{ut} is the user clock bias error. Besides the clock error, there are other factors affecting the pseudorange measurement. The measured pseudorange ρ_i can be written as

$$\rho_i = \rho_{iT} + \Delta D_i - c(\Delta b_i - b_{ut}) + c(\Delta T_i + \Delta I_i + v_i + \Delta v_i), \tag{4}$$

where ΔD_i is the satellite position error effect on the range, ΔT_i is the tropospheric delay error,

ΔI_i is the ionospheric delay error, v_i is the receiver measurement noise error, and Δv_i is the relativistic time correction. Some of these errors can be corrected for example, the tropospheric delay can be modeled and the ionospheric error can be corrected in a two-frequency receiver. The errors will cause inaccuracy of the user position. However, the user clock error cannot be corrected through receiver information. Thus, it will remain as an unknown. So, the system of (1) must be modified as

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + b_u, \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} + b_u, \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + b_u, \\
\rho_4 &= \sqrt{(x_4 - x_u)^2 + (y_4 - y_u)^2 + (z_4 - z_u)^2} + b_u,
\end{aligned} \tag{5}$$

where b_u is the user clock bias error expressed in distance, which is related to the quantity b_{ut} by $b_u = cb_{ut}$. In the system of (5), four equations are needed to solve four unknowns x_u, y_u, z_u and b_u. Thus, in a GPS receiver, a minimum of four satellites is required to solve the user position.

4. Solution of User Position from Pseudoranges

To solve the system of equation (5) is to linearize them. The system can be written in a simplified form as

$$\rho_i = \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2} + b_u, \, i = 1,2,3,4 \tag{6}$$

where x_u, y_u, z_u are unknowns. The pseudorange ρ_i and the positions of the satellites x_i, y_i, z_i are known. By differentiating (6), we have

$$\delta\rho_i = \frac{(x_i - x_u)\delta x_u + (y_i - y_u)\delta y_u + (z_i - z_u)\delta z_u}{\rho_i - b_u} + \delta b_u. \tag{7}$$

In (7), $\delta x_u, \delta y_u, \delta z_u, \delta b_u$ can be considered as the only unknowns. The quantities x_u, y_u, z_u, b_u are treated as known values because one can assume some initial values for these quantities. From these initial values, a new set of $\delta x_u, \delta y_u, \delta z_u, \delta b_u$ can be calculated. These values are used to modify the original x_u, y_u, z_u, b_u to find another new set of solutions. This new set of x_u, y_u, z_u, b_u can be considered again as known quantities. This process continues until the absolute values of $\delta x_u, \delta y_u, \delta z_u, \delta b_u$ are very small and within a certain predetermined limit.

The final values of x_u, y_u, z_u, b_u are the desired solution. This method is often referred to as an iteration method of fixed point. With $\delta x_u, \delta y_u, \delta z_u, \delta b_u$ as unknowns, the above equation becomes a set of linear equations. This procedure is often referred to as linearization. The expression (7) can be written in matrix form as

$$\begin{pmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \\ \delta\rho_4 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix} \begin{pmatrix} \delta x_u \\ \delta y_u \\ \delta z_u \\ \delta b_u \end{pmatrix}, \tag{8}$$

where

$$\alpha_{i1}=\frac{x_i-x_u}{\rho_i-b_u},\alpha_{i2}=\frac{y_i-y_u}{\rho_i-b_u},\alpha_{i3}=\frac{z_i-z_u}{\rho_i-b_u}. \tag{9}$$

The solution of (8) is

$$\begin{pmatrix}\delta x_u\\ \delta y_u\\ \delta z_u\\ \delta b_u\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & 1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 1\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1\\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1\end{pmatrix}^{-1}\begin{pmatrix}\delta\rho_1\\ \delta\rho_2\\ \delta\rho_3\\ \delta\rho_4\end{pmatrix}. \tag{10}$$

This process obviously does not provide the needed solutions directly. However, the desired solutions can be obtained from it. In order to find the desired position solution, this procedure must be used repetitively in an iterative way. A quantity is often used to determine whether the desired result is reached, and this quantity can be defined as

$$\delta v=\sqrt{\delta x_u^2+\delta y_u^2+\delta z_u^2+\delta b_u^2}, \tag{11}$$

where δv is lower than a certain predetermined threshold, the iteration will stop. Sometimes, the clock bias b_u is not included in (11).

5. Newton-Like solvers for solving the GPS navigation equations

The set of equation (6) produces a non-linear set of equations. In order to solve this nonlinear equations, the root function is first linearized according to the Taylor model. Expanding $F(x)$ about x_n gives

$$F(x)=F(x_n)+(x-x_n)F'(x_n)+\frac{1}{2!}(x-x_n)^2F''(x_n)+... \tag{12}$$

Calculation of second and higher order Frechet derivative ($F''(x_n),F'''(x_n),...$) may not be possible in some practical applications. For this reason we have omitted from the second term onwards in equation (12), then we have

$$F(x)=F(x_n)+(x-x_n)F'(x_n). \tag{13}$$

Putting $x=x_{n+1}$, $F(x_{n+1})\approx 0$ and by mathematical induction, the above equation can be written as

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n), n=0,1,2,.... \tag{14}$$

This is known as one-point classical Newton's method and its convergence order are two with two function evaluations. As we can verify from [11], the above iterative method is used to calculate, the receiver position in the GPS.

Traub [12] investigated that one-point iterative method of order p that they must depend explicitly on the first $p-1$ derivatives of f. This implies that their informational efficiency is less than or equal to unity and the higher order one-point iterative methods become increasingly complex by their formulae and the computational cost increases when higher derivatives of f are evaluated. He also proved that these drawbacks are absent for higher order multipoint iterative method. For this reason, great attention was paid to multipoint iterative methods since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. This class of methods was also extensively studied in many papers published in the recent past, see [1,2,3,8,10].

In this work, we have used the method of Noor et al. [8] to improve the GPS software by their efficient 2-step fourth order iterative method ($4^{th}\ NWNA$)

$$\begin{aligned} y_n &= x_n - [F'(x_n)]^{-1} F(x_n) \\ x_{n+1} &= y_n - [F'(y_n)]^{-1} F(y_n). \end{aligned} \tag{15}$$

Also, we have used the method of Abad et al. ($4^{th}\ ACT$) [1] which is given below

$$\begin{aligned} y_n &= x_n - [F'(x_n)]^{-1} F(x_n) \\ z_n &= x_n - [F'(x_n)]^{-1} (F(x_n) + F(y_n)) \\ x_{n+1} &= y_n - [F'(z_n)]^{-1} F(y_n). \end{aligned} \tag{16}$$

Further, we have used a sixth order method ($6^{th}\ BMJ$) given by Babajee et al. [2], another sixth order method ($6^{th}\ MJ$) by Madhu et al. [7] and a fifth order method ($5^{th}\ MBJ$) by Madhu et al. [6] respectively to improve the GPS software to determine the user position

$$\begin{cases} y_n = x_n - \frac{2}{3}[F'(x_n)]^{-1} F(x_n), \\ z_n = x_n - H_1(x_n) A(x_n) F(x_n), H_1(x_n) = I - \frac{1}{4}(\tau(x_n) - I) + \frac{1}{2}(\tau(x_n) - I)^2, \\ \tau(x_n) = [F'(x_n)]^{-1} F'(y_n), A(x_n) = \frac{1}{2}\left([F'(x_n)]^{-1} + [F'(y_n)]^{-1}\right), \\ x_{n+1} = z_n - H_2(x_n) A(x_n) F(z_n), H_2(x_n) = 2I - \tau(x_n). \end{cases} \tag{17}$$

$$\begin{cases} y_n = x_n - \frac{2}{3}[F'(x_n)]^{-1} F(x_n), \\ z_n = x_n - H_1(x_n)[F'(x_n)]^{-1} F(x_n), H_1(x_n) = I - \frac{3}{4}(\tau(x_n) - I) + \frac{9}{8}(\tau(x_n) - I)^2, \\ x_{n+1} = z_n - H_2(x_n)[F'(x_n)]^{-1} F(z_n), H_2(x_n) = I - \frac{3}{2}(\tau(x_n) - I) + \frac{1}{2}(\tau(x_n) - I)^2, \\ \tau(x_n) = [F'(x_n)]^{-1} F'(y_n). \end{cases} \tag{18}$$

$$\begin{cases} y_n = x_n - [F'(x_n)]^{-1} F(x_n), \\ x_{n+1} = y_n - H_1(x_n)[F'(x_n)]^{-1} F(y_n), \\ H_1(x_n) = 2I - \tau(x_n) + \frac{5}{4}(\tau(x_n) - I)^2, \quad \tau(x_n) = [F'(x_n)]^{-1} F'(y_n). \end{cases} \tag{19}$$

Where I represents the $n \times n$ identity matrix.

6. Efficiency of the Methods

A familiar tool to compare the efficiency of different iterative methods is the Efficiency Index (IE) which is defined by Ostrowski [9] as $IE = p^{\frac{1}{d}}$, where p is the order of convergence and d is the total number of function evaluations per iteration of the method. In [3], a combination of d and op is used to define the Computational Efficiency (CE) given as $CE = p^{1/(d+op)}$, where op is the total number of operations per iteration. We recall that the number of products and quotients required for solving m linear system with the same matrix of coefficient by using LU factorization is ($\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n$), where n is the size of each system. The IE and CE for $2^{nd}\ NM$, $4^{th}\ ACT$, $4^{th}\ NWNA$, $6^{th}\ BMJ$, $6^{th}\ MJ$ and $5^{th}\ MBJ$ are given in Table 1. It is observed from Fig. 3 that IE and CE for the $6^{th}\ MJ$ is better than $2^{nd}\ NR$. Also, $6^{th}\ MJ$ is having better CE index than other compared methods.

Table 1: Comparison of efficiency indices

Method	IE	CE
$2^{nd}\ NR$	$2^{\frac{1}{n+n^2}}$	$2^{\frac{1}{\frac{1}{3}n^3+2n^2+\frac{2}{3}n}}$
$4^{th}\ ACT$	$4^{\frac{1}{2n+2n^2}}$	$4^{\frac{1}{\frac{2}{3}n^3+5n^2+\frac{4}{3}n}}$
$4^{th}\ NWNA$	$4^{\frac{1}{2n+2n^2}}$	$4^{\frac{1}{\frac{2}{3}n^3+4n^2+\frac{4}{3}n}}$
$6^{th}\ BMJ$	$6^{\frac{1}{2n+2n^2}}$	$6^{\frac{1}{\frac{2}{3}n^3+6n^2+\frac{4}{3}n}}$
$6^{th}\ MJ$	$6^{\frac{1}{2n+2n^2}}$	$6^{\frac{1}{\frac{1}{3}n^3+5n^2+\frac{5}{3}n}}$
$5^{th}\ MBJ$	$5^{\frac{1}{2n+2n^2}}$	$5^{\frac{1}{\frac{1}{3}n^3+5n^2+\frac{5}{3}n}}$

7. Numerical Results for the GPS Problem

The numerical experiments have been carried out using the Matlab software for the examples given below. We use the following stopping criterion for the iteration scheme:

$$err_{min} = \| x^{(k+1)} - x^{(k)} \|_2 < 10^{-10}. \tag{20}$$

Let M be the number of iterations required for reaching the minimum residual (err_{min}).

In order to test the above listed iterative methods on the problem of a user position of a GPS device, we used the coordinates of observed satellite and pseudorange as calculated by El-naggar [4] and it is given in Table 2.

Table 2: Coordinates of observed satellite and pseudorange.

Sat #	X	Y	Z	ρ
SAT 1	17934700.08	-1201699.27	25412566.4	26063773.1
SAT 2	13642634.73	6241228.57	27327782.1	25880448.3
SAT 4	9078161.96	16940062.6	23258573.1	24898018.26
SAT 11	13950041.94	22815808.7	4876545.56	22162681.56
SAT 13	22247083.32	12695442.4	13764915.2	22716021.52
SAT 16	21944910.3	155,02,347	2490006.32	21452169.25

Table 3: Comparison of the iterative methods for GPS

Method	x_0	M	p_c	err_{min}	User position
2^{nd} NR	$(0,0,0,0)^T$	6	2.00	3.8641e-023	x^*
4^{th} ACT		4	3.99	9.1051e-032	x^*
4^{th} $NWNA$		4	4.06	1.4958e-031	x^*
6^{th} BMJ		4	5.64	7.5061e-015	x^*
6^{th} MJ		4	5.76	5.5526e-026	x^*
5^{th} MBJ		3	4.90	7.1806e-011	x^*

For our study, we need XYZ coordinates and Pseudorange (ρ) of only four satellites to find the user position. Hence, we consider data from four satellites namely SAT 1, SAT 2, SAT 4 and SAT 11. Table 3 compares different iterative methods for the nonlinear system used in the GPS software, where M represents the number iterations required for convergence. We recall that the coordinates of the center of the Earth and $b_u = 0$ gives $x_0 = (0,0,0,0)^T$ are usually used as starting value. We denote $x^* = (4732338.5\,12,\ 2723851.26\,8,\ 3285484.24\,0,\ 0.00001263\,5)^T$ as the solution of the nonlinear system which gives the user position on the Earth. We observe that all the iterative methods converge to the user position from the center of the earth. Among them methods tested, 5^{th} MBJ converges to the solution with less number of iterations than other comparable methods.

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