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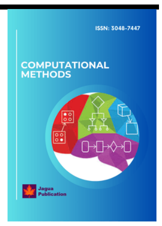

Research Article

Height and Fixed Points in *m*-Topological Transformation Semigroup Spaces

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Abstract

We investigate two enumerative invariants in three m-topological transformation semigroup spaces associated with full finite transformations, namely M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$. The first invariant is the *height* distribution, which classifies elements by image size. Using a sequence model over an extended alphabet, we reduce the height problem to classical surjection counting and derive explicit closed formulas via inclusion–exclusion; in the clopen setting the enumeration simplifies to a Stirling-number expression. The second invariant is the *fixed-point* distribution, which counts transformations (and complements) according to the number of fixed points. For each space we obtain closed counting formulas, establish supporting identities and recurrences, and present numerical tables for $1 \leq n \leq 8$. Illustrative plots for $n = 8$ show that height counts concentrate at intermediate image sizes, while fixed-point counts are largest for small numbers of fixed points, and they highlight how the closed and clopen restrictions shift these distributions relative to the full case.

1. Introduction

Transformation semigroups are fundamental objects in semigroup theory and finite combinatorics, serving as a tractable model in which structural and enumerative questions can be studied explicitly [1–3]. They also support a rich theory of combinatorial statistics, including displacement and fixed-point behaviour [4–6], as well as rank and generation phenomena in finite transformation settings [7, 8]. Let $X_n = \{1, 2, \ldots, n\}$. A partial transformation is a map $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to X_n$; it is *full* when $\mathrm{Dom}\,\alpha = X_n$ [9]. The full transformations form the classical semigroup T_n under composition, with $|T_n| = n^n$ [1, 3]. Motivated by this classical setting, Francis, Adeniji and Mogbonju introduced m-topological transformation semigroup spaces, where families of transformations are equipped with pointwise lattice-type operations and a complement operation compatible with topological considerations [10, 11]. Related "work"-type investigations in transformation semigroups and their variants further motivate the study of combinatorial invariants in such spaces [12–14].

In this paper we study two basic invariants for the full, closed and clopen m-topological spaces M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$: the *height* distribution (classification by image size) and the *fixed-point* distribution (classification by the number of fixed points), including fixed points of complements [5, 6]. To compute height distributions we use a sequence model over the extended alphabet $Y_n = \{0\} \cup X_n$ and count words by the number of distinct nonzero symbols. This converts the height problem into classical surjection enumeration, allowing systematic use of inclusion–exclusion and Stirling numbers [7, 15, 16]. We obtain explicit closed formulas for the height numbers in each of the three settings and derive closed expressions for fixed-point counts, supported by tables for small values of n.

2. Height of M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$

In this section we compute the height distributions (image-size distributions) for the full, closed and clopen m-topological transformation semigroup spaces M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$.

Example 2.1.

$$\textit{Case } k=0: \quad \Gamma(3;0)=1 \qquad \begin{pmatrix} 0 \\ (0,0,0) \end{pmatrix}$$

$$\textit{Case } k=1: \quad \Gamma(3;1)=14$$

For symbol 1*:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (1,1,1) \end{pmatrix}$$

For symbol 2*:*

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ (2,0,0) & (0,2,0) & (0,0,2) & (2,2,0) & (2,0,2) & (0,2,2) & (2,2,2) \end{pmatrix}$$

$$\textit{Case } k=2: \quad \Gamma(3;2)=12$$

$$\begin{pmatrix} 1,2 & 1,2 & 1,2 & 1,2 & 1,2 & 1,2 \\ (0,1,2) & (0,2,1) & (1,0,2) & (2,0,1) & (1,2,0) & (2,1,0) \end{pmatrix}$$

$$\begin{pmatrix} 1,2 & 1,2 & 1,2 & 1,2 & 1,2 & 1,2 \\ (1,1,2) & (1,2,1) & (2,1,1) & (2,2,1) & (2,1,2) & (1,2,2) \end{pmatrix}$$

Lemma 2.2. *Let $\alpha : X_n \to X_n$ be a full transformation and define $\alpha^c : X_n \to \{0,1,\dots,n-1\}$ by $\alpha^c(x) = n - \alpha(x)$. If $n \in \mathrm{Im}(\alpha)$, then $0 \in \mathrm{Im}(\alpha^c)$.*

Proof. If $n \in \mathrm{Im}(\alpha)$, there exists $x_0 \in X_n$ with $\alpha(x_0) = n$. Then $\alpha^c(x_0) = n - \alpha(x_0) = 0$, hence $0 \in \mathrm{Im}(\alpha^c)$. □

Theorem 2.3. *Let $\delta = M_{T_n}$. For $0 \le k \le n$,*

$$\Gamma(n,k) = \binom{n}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n,$$

and consequently

$$|M_{T_n}| = \sum_{k=0}^{n} \Gamma(n,k) = n^n.$$

Proof. Fix k and write $\Gamma(n,k) = \left|\{\alpha : X_n \to X_n : \ |\mathrm{Im}(\alpha)| = k\}\right|$. For each k-subset $Y \subseteq X_n$, let $A_Y = \{\alpha : X_n \to X_n : \ \mathrm{Im}(\alpha) = Y\}$. Then the sets $\{A_Y : |Y| = k\}$ form a partition of $\{\alpha : X_n \to X_n : \ |\mathrm{Im}(\alpha)| = k\}$,
Hence

$$\Gamma(n,k) = \sum_{\substack{Y \subseteq X_n \\ |Y|=k}} |A_Y| = \binom{n}{k} |A_{[k]}|,$$

since $|A_Y|$ depends only on $|Y| = k$.It remains to compute $|A_{[k]}|$, i.e. the number of surjections $f : X_n \to [k]$ is surjective. let $E_i = \{f : X_n \to [k] : \ i \notin \mathrm{Im}(f)\}$. Then the set of surjections is the complement of $\bigcup_{i=1}^{k} E_i$ inside $[k]^{X_n}$, so by inclusion–exclusion,

$$|A_{[k]}| = k^n - \left|\bigcup_{i=1}^{k} E_i\right| = \sum_{j=0}^{k} (-1)^j \sum_{\substack{J \subseteq [k] \\ |J|=j}} \left|\bigcap_{i \in J} E_i\right|.$$

If $J \subseteq [k]$ with $|J| = j$, then $\bigcap_{i \in J} E_i$ consists of maps whose images avoid the j symbols in J, hence maps from X_n into a set of size $k - j$, so $\left|\bigcap_{i \in J} E_i\right| = (k-j)^n$. There are $\binom{k}{j}$ such subsets J, therefore $|A_{[k]}| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$, and multiplying by $\binom{n}{k}$ yields the stated formula. □

Corollary 2.4. *For $0 \le k \le n$, the height numbers in M_{T_n} satisfy*

$$\Gamma(n,k) = \binom{n}{k} k!\, S(n,k),$$

where $S(n,k)$ denotes the Stirling number of the second kind.

Proof. The number of surjections from an n-element set onto a fixed k-element set is $k!\,S(n,k)$. In Theorem 2.3, $\Gamma(n,k)$ counts transformations with image size k, so we first choose the image set in $\binom{n}{k}$ ways and then count surjections onto that set, giving $\Gamma(n,k) = \binom{n}{k} k!\,S(n,k)$. □

Theorem 2.5. *For $1 \le k \le n$,*

$$\Gamma(n,k) = k\,\Gamma(n-1,k) + (n-k+1)\,\Gamma(n-1,k-1).$$

Proof. Using Corollary 2.4,

$$\Gamma(n,k) = \binom{n}{k} k!\,S(n,k).$$

The Stirling numbers satisfy the recurrence $S(n,k) = k\,S(n-1,k) + S(n-1,k-1)$. Therefore

$$\Gamma(n,k) = \binom{n}{k} k!\big(k\,S(n-1,k) + S(n-1,k-1)\big).$$

Now use the identities

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}, \qquad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1},$$

to rewrite:

$$\binom{n}{k} k! \cdot k\,S(n-1,k) = k \cdot \binom{n}{k} k!\,S(n-1,k) = k \cdot \frac{n}{n-k}\binom{n-1}{k} k!\,S(n-1,k),$$

and

$$\binom{n}{k} k!\,S(n-1,k-1) = (n-k+1)\binom{n-1}{k-1}(k-1)!\,S(n-1,k-1).$$

Recognizing $\Gamma(n-1,k) = \binom{n-1}{k} k!\,S(n-1,k)$ and $\Gamma(n-1,k-1) = \binom{n-1}{k-1}(k-1)!\,S(n-1,k-1)$ gives the stated recurrence. □

Theorem 2.6. *Let $\delta = M_{CT_n}$. For $0 \le k \le n$,*

$$\Gamma(n,k) = \begin{cases} \binom{n-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+1-j)^n, & 1 \le k \le n-1, \\ 1, & k = 0. \end{cases}$$

Proof. Fix $k \ge 1$. Choose a k-subset $Y \subseteq \{1,2,\dots,n-1\}$; there are $\binom{n-1}{k}$ choices. For such Y, consider the alphabet $A = \{0\} \cup Y$, so $|A| = k+1$. We count the sequences $\mathbf{a} = (a_1,\dots,a_n) \in A^n$ such that every symbol in Y appears at least once (while 0 may appear or not). For each $y \in Y$, let $E_y = \{\mathbf{a} \in A^n : y \text{ does not appear in } \mathbf{a}\}$. Then the admissible sequences are $A^n \setminus \bigcup_{y\in Y} E_y$, hence by inclusion–exclusion,

$$\left|A^n \setminus \textstyle\bigcup_{y\in Y} E_y\right| = \sum_{j=0}^{k} (-1)^j \sum_{\substack{J\subseteq Y \\ |J|=j}} \left|\textstyle\bigcap_{y\in J} E_y\right|.$$

If $J \subseteq Y$ has size j, then $\bigcap_{y\in J} E_y$ is the set of sequences over the alphabet $A \setminus J$, which has size $(k+1)-j$, so $\left|\bigcap_{y\in J} E_y\right| = (k+1-j)^n$. There are $\binom{k}{j}$ such subsets J, hence the number of admissible sequences for fixed Y equals $\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+1-j)^n$. Multiplying by $\binom{n-1}{k}$ gives the desired formula. For $k = 0$, the only sequence with no nonzero symbols is $(0,0,\dots,0)$, so $\Gamma(n,0) = 1$. □

Theorem 2.7. *Let $\delta = \mathrm{Clp}(M_{CT_n})$. For $0 \le k \le n$,*

$$\Gamma(n,k) = \binom{n-1}{k} k!\,S(n,k),$$

where $S(n,k)$ denotes the Stirling number of the second kind.

Proof. Fix k. Choose a k-subset $Y \subseteq \{1,2,\dots,n-1\}$; there are $\binom{n-1}{k}$ choices. For fixed Y, we count sequences in Y^n whose set of entries equals Y, i.e. surjections $f : [n] \to Y$. The number of surjections from an n-element set onto a k-element set is $k!\,S(n,k)$, since $S(n,k)$ counts partitions of $[n]$ into k nonempty blocks and $k!$ labels these blocks by the elements of Y. Multiplying by the number of choices of Y yields the formula. □

Table 1: Values of $\Gamma(n,k)) = M_{T_n}$ for $1 \leq n \leq 7$.

$n\backslash k$	1	2	3	4	5	6	7	8	$\sum\Gamma(n,k) = \lvert M_{T_n}\rvert$
1	1								1
2	2	2							4
3	3	18	6						27
4	4	84	144	24					256
5	5	300	1500	1200	120				3125
6	6	930	10800	23400	10800	720			46656
7	7	2646	63210	294000	352800	105840	5040		823543
8	8	7112	324576	2857680	7056000	5362560	1128960	40320	16777216

Table 2: Height of $\Gamma(n,k) = M_{CT_n}$ including the all-zero row (k=0).

$n\backslash k$	0	1	2	3	4	5	6	7	$\sum\Gamma(n,k) = \lvert M_{CT_n}\rvert$
1	1								1
2	1	3							4
3	1	14	12						27
4	1	45	150	60					256
5	1	124	1080	1560	360				3125
6	1	315	6020	21000	16800	2520			46656
7	1	762	28980	204120	378000	191520	20160		820543
8	1	1785	127050	1631700	5838840	6667920	2328480	181440	16777216

Table 3: Height of $\Gamma(n,k) = Clp(M_{CT_n})$

$n\backslash k$	1	2	3	4	5	6	7	$\sum\Gamma(n,k) = \lvert Clp(M_{CT_n})\rvert$
1	1							1
2	1							1
3	2	6						8
4	3	42	36					81
5	4	180	600	240				1024
6	5	620	5400	7800	1800			15625
7	6	1890	36120	126000	100800	15120		279936
8	7	5334	202860	1428840	2646000	1340640	141120	5764801

3. Fixed points in M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$

In this section we enumerate elements according to the number of fixed points, in the full, closed and clopen m-topological transformation semigroup spaces M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$.

Corollary 3.1. *Let $\delta = M_{T_n}$ and let $n \geq 1$. For $0 \leq m \leq n$, let $F(n;m)$ denote the number of transformations $\alpha \in \delta$ having exactly m fixed points. Then $F(n;m) = \binom{n}{m}(n-1)^{n-m}$. Consequently, $\sum_{m=0}^{n} F(n;m) = n^n$.*

Proof. Let $\mathrm{Fix}(\alpha) = \{i \in X_n : \alpha(i) = i\}$. We count maps $\alpha : X_n \to X_n$ with $|\mathrm{Fix}(\alpha)| = m$.

Choose the fixed-point set $F \subseteq X_n$ with $|F| = m$; there are $\binom{n}{m}$ choices. For each $i \in F$ we must have $\alpha(i) = i$. For each $j \in X_n \setminus F$ we require $\alpha(j) \neq j$, hence $\alpha(j)$ may be chosen arbitrarily from $X_n \setminus \{j\}$, giving $n-1$ possibilities. These choices are independent for distinct j, so for a fixed F there are $(n-1)^{n-m}$ maps. Multiplying yields the claimed formula. Summing over m counts all maps $X_n \to X_n$, hence equals n^n. □

Theorem 3.2. *Let $n \geq 1$ and let $\alpha = (a_1, \ldots, a_n) \in \{1, \ldots, n\}^n$. Define the complement by*

$$\alpha^c = (n-a_1,\, n-a_2,\, \ldots,\, n-a_n).$$

For $m \geq 0$, let $F(n;m)$ denote the number of mappings α such that α^c has exactly m fixed points (i.e. indices i satisfying $\alpha^c(i) = i$). Then

$$F(n;m) = \begin{cases} \binom{n-1}{m} n\,(n-1)^{n-1-m}, & 0 \leq m \leq n-1, \\ 0, & m \geq n. \end{cases}$$

Moreover, $\sum_{m=0}^{n-1} F(n;m) = n^n$.

Proof. For $i \in X_n$, the fixed-point condition $\alpha^c(i) = i$ is equivalent to $n - a_i = i \iff a_i = n - i$. If $i = n$ this forces $a_n = 0$, which is impossible since $a_n \in \{1, \ldots, n\}$. Hence n can never be a fixed point of α^c, so $m \le n - 1$.

Fix $m \in \{0, 1, \ldots, n-1\}$. Choose a set $F \subseteq \{1, 2, \ldots, n-1\}$ of size m to be the fixed-point set of α^c; there are $\binom{n-1}{m}$ choices. For each $i \in F$ the value a_i is forced to be $n - i$.

For each $i \in \{1, \ldots, n-1\} \setminus F$, the value $n - i$ is forbidden, while all other $n - 1$ values in $\{1, \ldots, n\}$ are allowed; thus there are $n - 1$ choices for each such i. Finally, at index n there is no restriction, so a_n has n choices. Therefore, for fixed F the number of admissible maps equals

$$n \cdot (n-1)^{(n-1)-m} = n\,(n-1)^{n-1-m}.$$

Multiplying by $\binom{n-1}{m}$ yields the stated formula. Summing over m counts all maps $X_n \to X_n$, hence equals n^n. □

Theorem 3.3. *Let $n \ge 2$. Consider all mappings $\alpha = (a_1, \ldots, a_n) : X_n \to \{1, 2, \ldots, n-1\}$ with complement $\alpha^c = (n - a_1,\, n - a_2,\, \ldots,\, n - a_n)$. For $m \ge 0$ let $F(n;m)$ denote the number of such mappings whose complement has exactly m fixed points. Then*

$$F(n;m) = \begin{cases} \binom{n-1}{m}(n-1)(n-2)^{n-1-m}, & 0 \le m \le n-1, \\ 0, & m \ge n, \end{cases}$$

and $\sum_{m=0}^{n-1} F(n;m) = (n-1)^n$.

Proof. For $i \in X_n$, the condition $\alpha^c(i) = i$ is equivalent to $a_i = n - i$. This is possible only for $i \in \{1, \ldots, n-1\}$, since $i = n$ would force $a_n = 0$, which is not allowed. Hence $m \le n - 1$.

Fix $m \in \{0, 1, \ldots, n-1\}$. Choose the fixed-point set $F \subseteq \{1, \ldots, n-1\}$ with $|F| = m$; there are $\binom{n-1}{m}$ choices. For each $i \in F$, the value a_i is forced to equal $n - i$.

For each $i \in \{1, \ldots, n-1\} \setminus F$, we must choose $a_i \in \{1, \ldots, n-1\} \setminus \{n-i\}$, which has size $n-2$, giving $n-2$ choices. At index n, all $n-1$ values are allowed, giving $n-1$ choices for a_n. Hence for fixed F the number of admissible maps equals $(n-1)(n-2)^{(n-1)-m}$. Multiplying by $\binom{n-1}{m}$ yields the stated formula. Finally,

$$\sum_{m=0}^{n-1} F(n;m) = (n-1)\sum_{m=0}^{n-1}\binom{n-1}{m}(n-2)^{n-1-m} = (n-1)(1+n-2)^{n-1} = (n-1)^n,$$

as required. □

Proposition 3.4. *Let $F(n;m) = \binom{n}{m}(n-1)^{n-m}$ be the number of maps $\alpha : X_n \to X_n$ with exactly m fixed points. Then for $1 \le m \le n$,*

$$F(n;m) = \frac{n-m+1}{m} F(n;m-1) \cdot \frac{1}{n-1}.$$

Equivalently,

$$\frac{F(n;m)}{F(n;m-1)} = \frac{n-m+1}{m(n-1)}.$$

Proof. Compute directly:

$$\frac{F(n;m)}{F(n;m-1)} = \frac{\binom{n}{m}(n-1)^{n-m}}{\binom{n}{m-1}(n-1)^{n-(m-1)}} = \frac{\binom{n}{m}}{\binom{n}{m-1}} \cdot \frac{1}{n-1}.$$

Using $\binom{n}{m}/\binom{n}{m-1} = (n-m+1)/m$ gives the claim. □

Table 4: Values of M_{T_n} $1 \le n \le 8$.

$n\backslash m$	0	1	2	3	4	5	6	7	8	$\sum_{m=0}^{n} F(n;m)$
1	0	1								1
2	1	2	1							4
3	8	12	6	1						27
4	81	108	54	12	1					256
5	1024	1280	640	160	20	1				3125
6	15625	18750	9375	2500	375	30	1			46656
7	279936	326592	163296	45360	7560	756	42	1		823543
8	5764801	6588344	3294172	941192	168070	19208	1372	56	1	16777216

Table 5: Values ofM_{CT_n}, $1 \le n \le 8$.

$n\backslash m$	0	1	2	3	4	5	6	7	8	$\sum_{m=0}^{n-1} F(n;m)$
1	1									1
2	2	2								4
3	12	12	3							27
4	108	108	36	4						256
5	1280	1280	480	80	5					3125
6	18750	18750	7500	1500	150	6				46656
7	326592	326592	136080	30240	3780	252	7			823543
8	6588344	6588344	2823576	672280	96040	8232	392	8		16777216

Table 6: Values ofClpM_{CT_n}, $1 \le n \le 8$.

$n\backslash m$	0	1	2	3	4	5	6	7	8	$\sum_{m=0}^{n-1} F(n;m)$
1										0
2	0	1								1
3	2	4	2							8
4	24	36	18	3						81
5	324	432	216	48	4					1024
6	5120	6400	3200	800	100	5				15625
7	93750	112500	56250	15000	2250	180	6			279936
8	1959552	2286144	1143072	317520	52920	5292	294	7		5764801

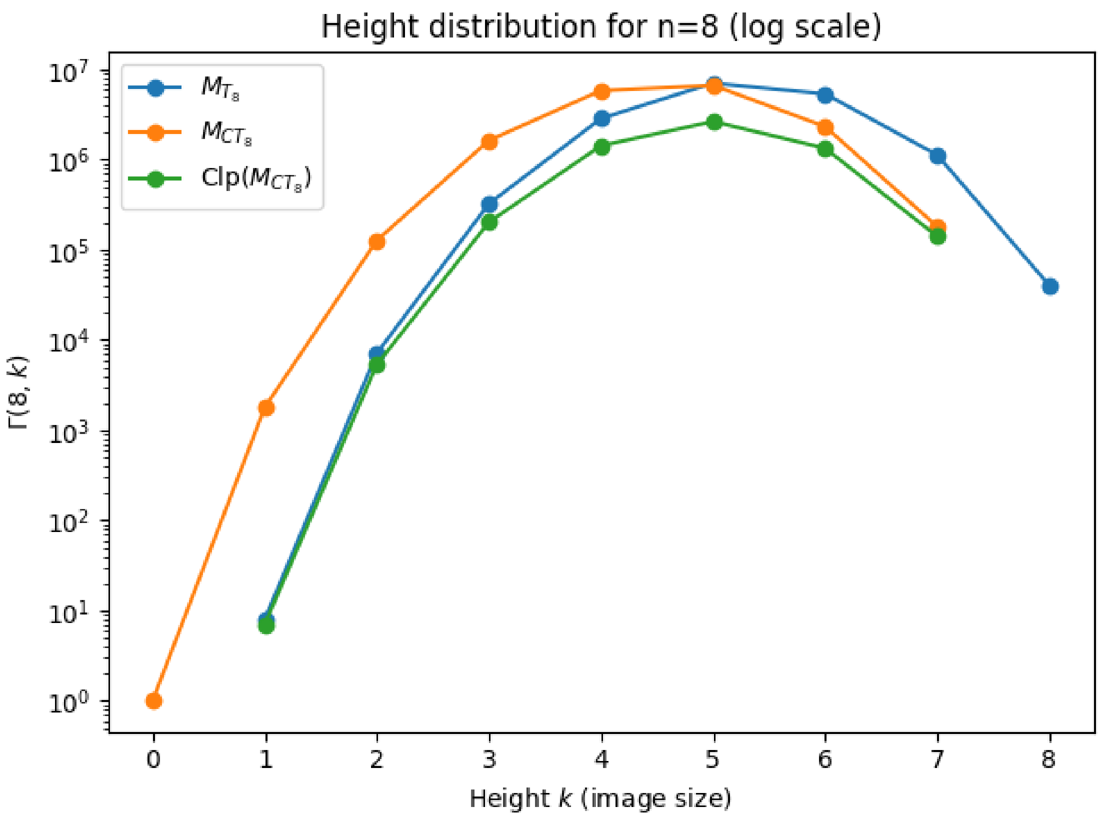

Figure 1: Height distributions $\Gamma(8,k)$ for M_{T_8}, M_{CT_8} and Clp(M_{CT_8}) (log scale).

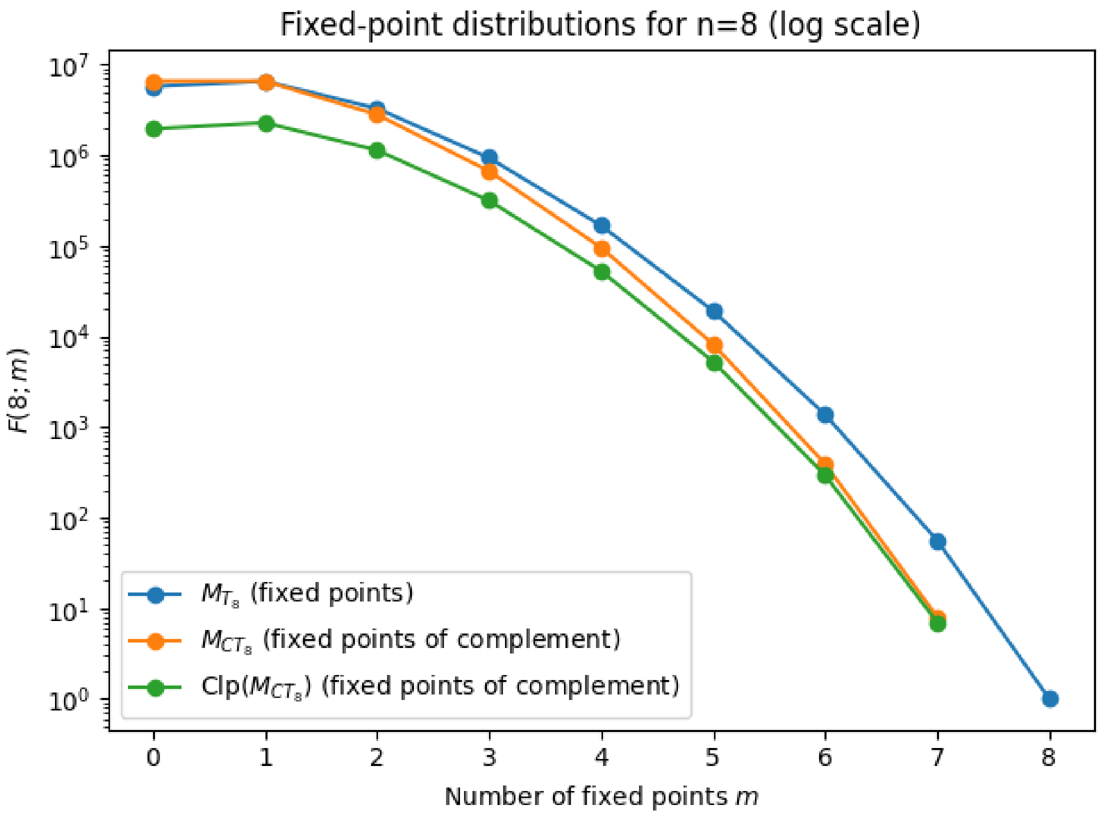

Figure 2: Fixed-point distributions $F(8;m)$ in M_{T_8} and fixed points of complements in M_{CT_8} and $\mathrm{Clp}(M_{CT_8})$ (log scale).

Figures 1 and 2 summarize the main enumerative behaviour of the three m-topological transformation semigroup spaces. Figure 1 compares the height distributions $\Gamma(8,k)$ and shows that most elements occur at intermediate image sizes, while very small or very large heights are relatively rare; the logarithmic scale highlights the wide variation in counts across k. Figure 2 presents the fixed-point distributions $F(8;m)$, indicating that transformations with few fixed points are most common and that the number of mappings decreases rapidly as m increases. Taken together, the figures visually support the closed-form formulas and show that the closed and clopen restrictions shift the distributions relative to the full case.

4. Conclusion

In this work we investigated two natural combinatorial invariants associated with the full, closed and clopen m-topological ransformation semigroup spaces M_{T_n}, M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$, namely the *height distribution* and the *fixed-point distribution* (classified by the number of fixed points). For the height invariant, we obtained explicit closed formulas for $\Gamma(n,k)$ in each setting. In the full case M_{T_n}, $\Gamma(n,k)$ is expressed as the number of surjections onto a k-set and is computed via inclusion–exclusion, recovering the classical identity $\sum_{k=0}^{n}\Gamma(n,k) = n^n$. In the closed case M_{CT_n}, the presence of the zero symbol leads to a modified inclusion–exclusion count on an alphabet of size $k+1$. In the clopen case $\mathrm{Clp}(M_{CT_n})$, the enumeration reduces to surjections onto k symbols and yields the compact Stirling form $\Gamma(n,k) = \binom{n-1}{k}k!\,S(n,k)$. The resulting tables illustrate the complete height distributions for small values of n and confirm the expected total sizes of the corresponding spaces. For the fixed-point invariant, we derived closed expressions for the number of transformations with exactly m fixed points in M_{T_n}, as well as the number of transformations whose *complements* have exactly m fixed points in M_{CT_n} and $\mathrm{Clp}(M_{CT_n})$. These formulas again sum to the correct total counts (n^n in the full/closed cases and $(n-1)^n$ in the clopen restriction), and the accompanying tables provide the full distributions for $1 \le n \le 8$. The formulas obtained here make the combinatorial structure of these m-topological transformation semigroup spaces explicit and reveal direct connections with classical tools such as inclusion–exclusion and Stirling numbers.

Article Information

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References

[1] J. M. Howie. *Fundamentals of Semigroup Theory*. Oxford University Press, 1995.

[2] A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups, Vol. I.* American Mathematical Society, 1961.

[3] O. Ganyushkin and V. Mazorchuk. *Classical Finite Transformation Semigroups*. Springer, 2009.

[4] D. Daly and P. Vojtěch. How permutations displace points and stretch intervals. *Ars Combinatoria*, 90:175–191, 2009.

[5] D. Daly. Fixed points and displacement in transformation semigroups. *Semigroup Forum*, 79:403–415, 2009.

[6] A. Umar and P. M. Higgins. Fixed points and idempotents in semigroups of partial transformations. *Communications in Algebra*, 24(1): 117–132, 1996.

[7] L. Comtet. *Advanced Combinatorics*. Reidel, 1974.

[8] G. M. S. Gomes and J. M. Howie. On the ranks of certain semigroups of order-preserving transformations. *Semigroup Forum*, 45: 272–282, 1992.

[9] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences. https://oeis.org/. Accessed 2026-02-25.

[10] M. O. Francis, A. O. Adeniji, and M. M. Mogbonju. Work done by m-topological transformation semigroup regular spaces m_{ψ_n}. *International Journal of Mathematical Sciences and Optimization Theory and Applications*, 9(1):33–42, 2023. doi: 10.6084/zenodo.8217976.

[11] M. O. Francis, A. O. Adeniji, and M. M. Mogbonju. Operation and vector spaces on m-topological transformation semigroup. *Journal of Linear and Topological Algebra*, 12(2):133–140, 2023. doi: 10.30495/jlta.2023.704265.

[12] M. O. Francis and A. O. Adeniji. Work performed by closed and clopen m-topological full transformation semigroup spaces mctn and clp (mctn). *Unilag Journal of Mathematics and Applications*, 5(1):51–62, 2025.

[13] R. Kehinde and A. O. Habib. Numerical solutions of the work done on finite order-preserving injective partial transformation semigroup. *International Journal of Innovative Science and Research Technology*, 5(9):2456–2165, 2020.

[14] A. Umar. Some combinatorial problems in the theory of partial transformation semigroups. *Algebra and Discrete Mathematics*, pages 1–26, 2014.

[15] J. Riordan. *An Introduction to Combinatorial Analysis*. Dover Publications, 2002.

[16] R. P. Stanley. *Enumerative Combinatorics, Vol. 1*. Cambridge University Press, 2012.